On the Efficiency of Equilibria in Mean-field Oscillator Games

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October 16, 2013

Abstract

A key question in the design of engineered competitive systems has been that of the efficiency loss of the associated equilibria. Yet, there is little known in this regard in the context of stochastic dynamic games, particularly in a large population regime. In this paper, we revisit a class of noncooperative games, arising from the synchronization of a large collection of heterogeneous oscillators. In [31], we derived a PDE model for analyzing the associated equilibria in large population regimes through a *mean field approximation*. Here, we examine the efficiency of the associated mean-field equilibria with respect to a related welfare optimization problem. We construct constrained variational problems both for the noncooperative game and its centralized counterpart and derive the associated nonlinear eigenvalue problems. A relationship between the solutions of these eigenvalue problems is observed and allows for deriving an expression for efficiency loss. By applying bifurcation analysis, a local bound on efficiency loss is derived under an assumption that oscillators share the same frequency. Through numerical case studies, the analytical statements are supported in the homogeneous frequency regime; analogous numerical results are provided for the heterogeneous frequency regime.

1 Introduction

This paper concerns control synthesis and analysis for complex systems, described by a large population of coupled heterogeneous nonlinear stochastic systems. A central goal is to estimate the efficiency loss for large systems. Applications appear in many settings, such as economics, neurobiology, and telecommunications [11, 10, 9, 25, 28, 23]. Except in the simplest cases, it is impossible to obtain exact solutions to the dynamic programming equations to obtain optimal control solutions. This barrier is well known, and is the motivation for a large literature on approximation techniques for complex networked systems. The heavy-traffic theory developed in queueing theory is an example of optimal control approximation in a dynamic setting [21]. Often in such settings, agents have conflicting objectives, which motivates game-theoretic counterparts of such multi-agent systems. Expectedly, obtaining expressions for the equilibria

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arising from the resulting dynamic games is no less challenging. Yet, by employing a mean-field approximation, control synthesis has proved possible in some regimes [9, 8, 29, 31]. More recently, there have been efforts to examine risk-based [27] and robust [2] generalizations.

An immediate question is whether the equilibria associated with such mean-field approximations are indeed optimal (or near-optimal) solutions to the original problem of controlling a large population of nonlinear stochastic systems. Such a claim is seen to hold if the equilibria are efficient or that the equilibria lead to no loss in social welfare. The analysis of efficiency loss in complex finite-player *static* games has been studied extensively in the computer science and operations research community, with a focus on routing [23, 22, 18, 24, 13, 3, 4], resource allocation [15, 14, 16, 2], power systems [19, 20, 6, 5], and markets [1]. It is not obvious how these techniques can be directly extended to the dynamic and stochastic models considered here.

The present paper concerns approximation techniques for both centralized optimal control and dynamic games based upon the theory of mean-field games. The approximation techniques are based on an asymptotic setting, in which the number of players tends to infinity, extending the mean-field game theory of [9, 17, 29]. This approach is similar to the mean-field approximation techniques that form one foundation of statistical mechanics [17]. In this setting, the goal is to understand the aggregate behavior of a large number of interacting particles. Under appropriate conditions, as the number of particles tends to infinity, the fluctuations of the mass influence on an individual particle are “averaged out” [17, 9, 29]. Consequently, this allows for any agent to make decisions based on its state and the mass influence, which is typically shown to be deterministic. Together with a consistency requirement imposed by the mass-influence, the resulting problem in an infinite population setting can be reduced to a set of coupled PDEs. There has been relatively little effort applied towards the quantification of efficiency loss in the mean-field regime. Huang et al. [9] provide an efficiency bound derived from comparing the infinite horizon costs between the centralized control and their decentralized mean-field counterparts in a setting with quadratic costs and linear dynamics.

The analysis in the present paper is focused on a particular model of coupled oscillators introduced in [31, 33] where the synchronization of a collection of oscillators, a stochastic control problem, is modeled as a dynamic game. In the infinite-population models considered in this paper, the solutions of this centralized control problem and the equilibria of the associated dynamic game correspond to the solutions to two sets of coupled PDEs. An analysis of the two models in an infinite population regime leads to an expression for the efficiency loss associated with the game. This exact analysis is possible in a special case in which the population is homogeneous, in the sense that each oscillator shares a common natural frequency. Succinctly, the main contributions of this paper pertain to the analysis of the efficiency of equilibria associated with the stochastic dynamic games:

(i) The mean-field approximations for the centralized stochastic control problem and its game-theoretic counterpart are both represented as stochastic variational problems, each of which is represented as a nonlinear eigenvalue problem.

(ii) A precise relationship is established between the two variational problems, which leads to a simple expression for efficiency loss. Furthermore, under a regime in which all oscillators share the same frequency (homogeneous regime), locally valid bounds on efficiency loss are obtained through bifurcation analysis.

(iii) These theoretical results for the infinite-population model are supported through numerical solutions in the homogeneous and heterogeneous regimes.

Before proceeding, the challenge in deriving efficiency statements imposes some limitations on the generality of the results. Specifically, the analytical statements are restricted to a class of solutions and the obtained efficiency bounds are in such a regime. The remainder of the paper is organized as follows. Background regarding the oscillator model, and the optimization problems considered in this paper appears in Sec. 2. The two variational problems are introduced in Sec. 3, whose solutions characterize system behavior in the infinite population limit. This paves the way for an efficiency analysis of the equilibria associated with the game-theoretic problem in Sec. 4. Numerical results are presented in Sec. 5, and conclusions in Sec. 6.

2 Preliminaries and problem statement

2.1 Oscillator game

The oscillator game model is comprised of a population of N oscillators. The model for the i^{th} oscillator is given by,

$$\mathrm{d}\theta_i(t) = (\omega_i + u_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i(t), \quad \mod 2\pi, \tag{1}$$

where $\theta_i(t) \in [0, 2\pi]$ is the phase of the i^{th} oscillator at time t, ω_i is the natural frequency, $u_i(t)$ is the control input, and $\{\xi_i(t),\ i = 1, \dots, N\}$ are mutually independent standard Wiener processes.

The control problem is a game: Specifically, the i^{th} oscillator seeks to minimize its own performance objective, given the decisions of (competing) oscillators:

$$\eta_i^{(\text{POP})}(u_i; u_{-i}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{E}[c(\theta_i; \theta_{-i}) + \tfrac{1}{2}Ru_i^2]\,\mathrm{d}s, \tag{2}$$

where $\theta_{-i} = (\theta_j)_{j\neq i}$, $c(\cdot)$ is a cost function, $u_{-i} = (u_j)_{j\neq i}$ and R is the control penalty parameter. The form of the function c and the value of R are assumed to be common to the entire population.

The $N-$player noncooperative game is denoted by $\mathcal{G}_N$. A *Nash equilibrium* is defined as a tuple of control policies $(u_i)_{i=1}^N$ such that u_i

$$\text{minimizes } \eta_i^{(\text{POP})}(u_i; u_{-i}), \quad i = 1, \dots, N. \tag{3}$$

Motivated in part by [8], we also consider a centralized *welfare optimization problem*, denoted by $\mathcal{W}_N$. The objective here is to chose the control input vector $u = (u_i)_{i=1}^N$ to minimize the welfare,

$$\eta^{(\text{OPT})}(u) \triangleq \frac{1}{N}\sum_{i=1}^N \eta_i^{(\text{POP})}(u_i; u_{-i}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^N \left[c(\theta_i; \theta_{-i}) + \tfrac{1}{2}Ru_i^2\right]\,\mathrm{d}s. \tag{4}$$

One of the objectives of the present paper is to characterize the *loss in efficiency* in going from the (centralized) welfare problem to the (distributed) game theoretic problem: Suppose u is an equilibrium of $\mathcal{G}_N$, then it is said to be an *efficient* equilibrium if $\eta^{(\mathrm{OPT})}(u) = N^{-1}\sum_{i=1}^{N}\eta_i^{(\mathrm{POP})}$; if not, then the loss in efficiency is defined as follows:

$$\Delta_\eta^N \triangleq \frac{1}{N}\sum_{i=1}^{N}\eta_i^{(\mathrm{POP})} - \eta^{(\mathrm{OPT})}(u).$$

For analytical tractability, we will consider the infinite-N limit. Specifically, we will derive certain formulae for loss of efficiency for $\mathcal{G}_\infty$ with respect to $\mathcal{W}_\infty$.

There are several functional analytic subtleties concerning the existence of a unique solution in the appropriate function space. While the question of uniqueness is not a focus of this paper, we assume that all minimization/infimization problems have a unique solution. The following assumption is employed in the remainder of this paper:

Assumption 1. (i) *For each i, $\{(\theta_i(0),\omega_i)\}$ is i.i.d., independent of $\{\xi_i\}$, with common marginal distribution $(\theta_i(0),\omega_i) \sim p(\theta,0;\omega)g(\omega)$.*

The frequency ω_i is a constant independent of time — It is assumed that at time $t = 0$, the N scalars $\{\omega_i\}$ are chosen independently according to a fixed uniform distribution *with density $g(\omega) = \frac{1}{2\gamma}$, which is supported on an interval of the form $\Omega = [1-\gamma, 1+\gamma]$ where $\gamma < 1$ is assumed to be a small constant. For a* homogeneous population *$\gamma = 0$ and $g(\omega) = \delta(\omega - 1)$, the Dirac delta function at $\omega = 1$.*

(ii) *The cost function c is separable, as shown below*

$$c(\theta_i;\theta_{-i}) := \frac{1}{N}\sum_{j\neq i} c^\bullet(\theta_i - \theta_j(t)), \tag{5}$$

where $c^\bullet$ is assumed to be a bounded function that satisfies the following properties:

- *(i) $c^\bullet$ is spatially invariant, i.e., $c^\bullet(\vartheta,\theta) = c^\bullet(\vartheta - \theta)$,*
- *(ii) $c^\bullet$ is 2π-periodic, i.e., $c^\bullet(\vartheta - \theta) = c^\bullet(\vartheta - \theta + 2\pi)$,*
- *(iii) $c^\bullet$ is non-negative, i.e., $c^\bullet(\vartheta - \theta) \geq 0$,*
- *(iv) $c^\bullet$ is an even function, i.e., $c^\bullet(\vartheta - \theta) = c^\bullet(\theta - \vartheta)$.*

In a numerical example, we take $c^\bullet(\theta - \vartheta) = \frac{1}{2}\sin^2\left(\frac{\theta-\vartheta}{2}\right)$.

(iii) *We assume that there exist unique solutions to all infimization/minimization problems.*

The considerations of this paper are based on an infinite-population limit similar to those introduced in our prior work [31, 33] and by others (e.g., [26]): We construct a density function p that is intended to approximate the probability density function for the individual oscillators. For any i and any $t > 0$, the density $p(\theta, t; \omega_i)$ is intended to approximate the probability density of the random variable $\theta_i(t)$, evolving according to the stochastic differential equation (1).

For a generic oscillator with frequency ω and control u, the density p is a 2π-periodic function in θ-variable, that evolves according to the Fokker-Planck-Kolmogorov (FPK) equation:

$$\partial_t p + \partial_\theta \left((\omega + u)p\right) = \frac{\sigma^2}{2}\partial^2_{\theta\theta} p\,, \tag{6}$$

where ∂_t and ∂_θ denote the partial derivative with respect to t and θ, respectively, and $\partial^2_{\theta\theta}$ denotes the second derivative with respect to θ.

In this paper, a certain steady-state traveling wave solution will be of interest. For such a solution, the following Lemma characterizes a useful relationship between control u and density p. The proof appears in the Appendix.

Lemma 1. *Suppose p is a 2π-periodic positive solution of the FPK equation (6), of the traveling wave form:*

$$p(\theta, t; \omega) = p(\theta - at, 0; \omega),$$

where $a \in \Re$. Then the control input u is related to the density p by,

$$u = \frac{\sigma^2}{2}\partial_\theta \ln p + \left(1 - \frac{2\pi}{p\int_0^{2\pi}\frac{1}{p}\,\mathrm{d}\theta}\right)(a - \omega). \tag{7}$$

■

Remark 1. *The mean-field oscillator game model described here was introduced in [31, 33]. Although the original model in these papers is posed as a game, one could also consider the welfare optimization problem as a starting point. Design of distributed control laws for such problems is challenging. One avenue is to allow agents to compete in a distributed fashion, and this in turn provides motivation for the game theoretic problem. The loss of efficiency may be regarded as a measure of performance degradation in going from the welfare to the game problem.*

2.2 Mean-field approximation

The control for the game problem is based on a mean-field approximation in the infinite-N limit. Specifically, the density $p(\theta, t; \omega)$ is used to define,

$$\bar{c}(\theta, t) := \int_\Omega \int_0^{2\pi} c^\bullet(\theta - \vartheta)p(\vartheta, t; \omega)g(\omega)\,\mathrm{d}\vartheta\,\mathrm{d}\omega. \tag{8}$$

The law of large numbers suggests the approximation of $c(\theta; \theta_{-i}(t))$ by $\bar{c}(\theta, t)$, when N is large.

For the scalar model (1) with cost $\bar{c}(\theta, t)$ depending only on $\theta_i = \theta$, the game reduces to independent optimal control problems. The associated average-cost HJB equation is given by,

$$\min_{u_i}\{\bar{c}(\theta, t) + \tfrac{1}{2}Ru_i^2 + \mathcal{D}_{u_i}h_i\,(\theta, t)\} = \eta_i^*, \tag{9}$$

where h_i denotes the relative value function of the ith agent, and $\mathcal{D}_u$ denotes the controlled generator, defined for C^2 functions f via,

$$\mathcal{D}_u f = \partial_t f + (\omega_i + u)\partial_\theta f + \frac{\sigma^2}{2}\partial^2_{\theta\theta} f.$$

Because the cost is quadratic in u_i and the dynamics are linear in u_i, this leads to the HJB equation

$$\partial_t h_i + \omega \partial_\theta h_i = \frac{1}{2R}(\partial_\theta h_i)^2 - \bar{c}(\theta, t) + \eta_i^* - \frac{\sigma^2}{2}\partial_{\theta\theta}^2 h_i,$$

and the optimal control law

$$u_i^* = -\frac{1}{R}\partial_\theta h_i(\theta_i, t). \tag{10}$$

2.3 PDE model

The mean-field game model is given by a coupled set of partial differential equations (PDEs) for the relative value function $h(\theta, t; \omega)$ and the density $p(\theta, t; \omega)$:

$$\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta, t) + \eta^* - \frac{\sigma^2}{2}\partial_{\theta\theta}^2 h, \tag{11a}$$

$$\partial_t p + \omega \partial_\theta p = \frac{1}{R}\partial_\theta \left[p(\partial_\theta h)\right] + \frac{\sigma^2}{2}\partial_{\theta\theta}^2 p, \tag{11b}$$

$$\bar{c}(\theta, t) = \int_\Omega \int_0^{2\pi} c^\bullet(\theta - \vartheta) p(\vartheta, t; \omega) g(\omega)\, \mathrm{d}\vartheta\, \mathrm{d}\omega. \tag{11c}$$

The PDE require initial and boundary condition, e.g., $(p, h, \bar{c})$ are assumed to be 2π-periodic in θ-variable; cf., [33]. The coupled PDE model is referred to as the *MF-HJB equation.*

2.4 ϵ-Nash equilibrium

Suppose $\{p(\theta, t; \omega), h(\theta, t, \omega)\}$ is a time-independent or time-periodic smooth solution of (11a)-(11c). For a finite population, each oscillator is controlled using the control solution in (10),

$$u_i^o = -\frac{1}{R}\partial_\theta h(\theta_i(t), t; \omega)\Big|_{\omega=\omega_i}.$$

Thm. 3.3 in [33], repeated below, shows that this control law is an ϵ-Nash equilibrium for (2).

Theorem 1. *For large N, the oblivious control $\{u_i^o\}$ is an ϵ-Nash equilibrium for (2): For any admissible control u_i,*

$$\eta_i^{(POP)}(u_i^o; u_{-i}^o) \leq \eta_i^{(POP)}(u_i; u_{-i}^o) + \epsilon_N,$$

where $\epsilon_N \to 0$ as $N \to \infty$ a.s., and in mean square with rate N^{-1}.

2.5 Special solutions

An ϵ-Nash equilibrium of the game (1) - (5) is obtained by considering the solution of the PDEs (11a) - (11c). Two types of solution are described in [33]:

(i) *Incoherence solution:*

$$p(\theta,t;\omega) = \frac{1}{2\pi},\ h(\theta,t;\omega) = 0\,,$$

with associated control law $u(t) \equiv 0$.

(ii) *Synchrony solution:* The traveling wave equation,

$$p(\theta,t;\omega) = p(\theta - t, 0;\omega),\ h(\theta,t;\omega) = h(\theta - t, 0;\omega),$$

where the population moves with a constant wave speed, 1, along the circle $[0, 2\pi]$. The distribution p is uni-modal and positive.

In this paper, with a slight abuse of notation, $p(\theta;\omega)$ is used to denote $p(\theta, 0;\omega)$, and similarly $h(\theta;\omega) := h(\theta, 0;\omega)$. Note that the traveling wave solution is obtained simply by rotating this solution with a constant wave speed, i.e., $p(\theta,t;\omega) = p(\theta - at;\omega)$ and $h(\theta,t;\omega) = h(\theta - at;\omega)$.

3 Variational problems

Bounding the efficiency loss in the mean-field regime is generally difficult since it necessitates obtaining an expression for the mean-field equilibrium and subsequently deriving bounds using the associated welfare problem. Instead, we show in Section 3.2, that the solution of a single optimization problem provides us with precisely the mean-field equilibrium of interest. In Section 3.3, we formulate a variant of this problem whose solution is shown to solve the centralized welfare problem. Together, these two variational problems provide the basis for conducting the efficiency analysis. Notably, the choice of these problems is by no means arbitrary and is closely tied to the Euler-Lagrange conditions associated with the game-theoretic and optimization problems, respectively. We begin by introducing the notations and assumptions for the mean-field limit.

3.1 A stationary mean-field model

It is assumed that for each N and i, the system (1) is controlled using a (possibly time varying) Markov policy. That is, $u_i(t)$ is a function of $\{\theta_j(t) : 1 \le j \le N\}$, N, i, and t. To characterize the set of possible equilibria for the limiting model is beyond the scope of this work. Instead, we restrict attention to either time-independent or periodic solutions for the infinite-N limit. The motivation comes from the incoherence and the synchrony solutions described in Sec. 2.5.

The following assumption will be in place throughout this section.

Assumption 2. (i) *A mean-field density p is obtained in the infinite-N limit: For any bounded measurable function $\chi\colon [0,2\pi]\times\Omega \to \Re$, and each $t \ge 0$, the following limit exists a.s.,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \chi(\theta_j(t),\omega_j) = \frac{1}{2\gamma}\int_{\omega=1-\gamma}^{1+\gamma}\int_{\vartheta\in[0,2\pi]} \chi(\vartheta,\omega)p(\vartheta,t;\omega)g(\omega)\ \mathrm{d}\vartheta\ \mathrm{d}\omega\,.$$

(ii) *The density $p(\theta, t; \omega)$ is positive and either time-independent or periodic in t. The periodic solution can be expressed as a traveling wave with wave speed a. As before, $p(\theta; \omega)$ is used to denote $p(\theta, 0; \omega)$.*

□

Observe that Assumption 2 (ii) presumes that any time-dependent mean-field limit is in a traveling wave steady-state. This is imposed because synchrony solution is a traveling wave solution with wave-speed $a = 1$. Under this assumption, the mean-field cost

$$\bar{c}(\theta, t) := \int_{\Omega} \int_0^{2\pi} c^{\bullet}(\theta - \vartheta) p(\vartheta - at; \omega) g(\omega) \, \mathrm{d}\vartheta \, \mathrm{d}\omega. \tag{12}$$

is periodic, and we can expect periodic solutions to the MF-HJB equation. In fact, $\bar{c}(\theta, t) = \bar{c}(\theta - at, 0)$, and is simply denoted as $\bar{c}(\theta - at)$.

Also, under these assumptions, the ergodic steady-state distribution of a generic oscillator, $\{\theta(t)\}$ with frequency ω, exists and moreover,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} c^{\bullet}(\theta(t) - \theta_j(t)) \, \mathrm{d}t = \lim_{T\to\infty} \frac{1}{T} \int_0^T \bar{c}(\theta(t) - at) \, \mathrm{d}t = \int_0^{2\pi} p(\theta; \omega) \bar{c}(\theta) \, \mathrm{d}\theta.$$

3.2 Variational formulation of $\mathcal{G}_\infty$

Let $\mathbf{X} := C^2_{2\pi}([0, 2\pi], \mathbb{R}^+)$ denote the space of twice continuously differentiable nonnegative real-valued periodic functions on $[0, 2\pi]$. We consider the following variational problem:

$$\begin{aligned} \min_{v\in\mathbf{X}} \quad \eta^{\mathrm{g}}(v; \bar{c}, \omega, a) : = \int_0^{2\pi} \Bigg[& \bar{c}(\theta) v^2 + \frac{R\sigma^4}{2} (\partial_\theta v)^2 \\ & + \frac{R}{2} (\omega - a)^2 v^2 \left(1 - \frac{2\pi}{v^2 \int_0^{2\pi} v^{-2} \, \mathrm{d}\theta} \right)^2 \Bigg] \mathrm{d}\theta, \end{aligned} \tag{13}$$

$$\text{s.t.} \quad 1 = \int_0^{2\pi} v^2(\theta; \omega) \, \mathrm{d}\theta. \tag{14}$$

The necessary conditions of optimality of this problem are captured by the Euler-Lagrange conditions, stated below and proved in the Appendix 7.2.

Lemma 2. *Suppose v is a critical point of (13)-(14) given the parameters ω, a, and the function $\bar{c}(\theta)$. Then, there exists a λ such that (v, λ) is a solution of*

$$\partial^2_{\theta\theta} v + \frac{2}{R\sigma^4} (\lambda - \bar{c}) v - \frac{(\omega - a)^2}{\sigma^4} \left(1 - \left(\frac{2\pi}{v^2 \int_0^{2\pi} v^{-2} \, \mathrm{d}\theta} \right)^2 \right) v = 0, \tag{15}$$

$$\int_0^{2\pi} v^2(\theta; \omega) \, \mathrm{d}\theta = 1, \tag{16}$$

where λ is the Lagrange multiplier associated with the constraint (14).

Our interest lies in the *constrained variational problem*; specifically, we are interested in critical points to (13)-(14) satisfying the additional requirement

$$\bar{c}(\theta) = \int_\Omega \int_0^{2\pi} c^\bullet(\theta - \vartheta) v^2(\vartheta;\omega) g(\omega)\,\mathrm{d}\vartheta\,\mathrm{d}\omega =: \mathcal{C}[v](\theta). \tag{17}$$

We denote this constrained variational problem as $\mathrm{V}_\infty^{\mathcal{G}}$.

From Lemma 2, we obtain the necessary conditions of optimality. The Lagrange multiplier is seen to give the optimal value of the variational problem. In the following, $\mathbf{V}$ denotes the subspace of functions $v(\in \mathbf{X})$ that satisfies the density constraint (14).

Lemma 3. *For any given value of parameters ω and a, suppose $(\bar{c}^*, v^*)$ is a solution of $V_\infty^{\mathcal{G}}$, corresponding to (13)-(14), (17), and λ^* is the corresponding Lagrange multiplier. Then $(\bar{c}^*, v^*, \lambda^*)$ is a solution of the problem (15)-(17) and $\lambda^*(\omega, a) = \eta_g^*(\omega, a) := \eta^g(v^*; \bar{c}^*, \omega, a) = \min_{v\in\mathbf{V}} \eta^g(v; \bar{c}^*, \omega, a)$.*

Next, we derive the main result of this subsection: a MFE of $\mathcal{G}_\infty$ is a minimizer of $\mathrm{V}_\infty^{\mathcal{G}}$. The nonnegativity of the density p allows for constructing a v such that $(v)^2 = p$.

Theorem 2. *Suppose $(h^*, (v^*)^2, \eta^*)$ is a MFE of the dynamic game $\mathcal{G}_\infty$, and $\bar{c}^*$ is the mass-influence function given by (11c). Under Assumption 2, $(\bar{c}^*, v^*)$ is a minimizer of $V_\infty^{\mathcal{G}}$.*

Proof. We begin by recalling the optimization problem for the i^{th}-oscillator: (18), given by

$$\eta_i^{(\text{POP})}(u_i; u_{-i}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [c(\theta_i; \theta_{-i}) + \tfrac{1}{2} R u_i^2]\,\mathrm{d}s. \tag{18}$$

We drop the subscript i, denoting $\theta_i(t)$ by $\theta(t)$ and frequency ω_i as ω.

The mean-field approximation together with the traveling wave assumption (see Assumption 2) are the key to simplify the two terms in the integrand:

(i) In the infinite-N limit, $c(\theta; \theta_{-i}(t))$ is approximated by $\bar{c}(\theta, t)$ (see (12)). Using Assumption 2, the approximation is expressed as $\bar{c}(\theta - at)$.

(ii) From Lemma 1, the relationship between the control u and the density p is captured by (7); specifically, $u = \frac{\sigma^2}{2}\partial_\theta \ln p + \left(1 - \frac{2\pi}{p\int_0^{2\pi}\frac{1}{p}\,\mathrm{d}\theta}\right)(a - \omega)$.

Substituting this in (18), we obtain the approximation of $\eta_i^{(\text{POP})}$ as

$$\begin{aligned}
\eta(p; \bar{c}, \omega, a) = &\underbrace{\lim_{T\to\infty} \frac{1}{T}\int_0^T \left[\bar{c}(\theta(s) - as) + \frac{R\sigma^4}{8}(\partial_\theta \ln p)^2(\theta(s) - as; \omega)\right]\mathrm{d}s}_{=:I_1} \\
&+ \underbrace{\lim_{T\to\infty} \frac{1}{T}\int_0^T \left[\frac{R}{2}(\omega - a)^2 \left(1 - \frac{2\pi}{p\int_0^{2\pi} p^{-1}\,\mathrm{d}\theta}\right)^2\right]\mathrm{d}s}_{=:I_2} \\
&+ \underbrace{\lim_{T\to\infty} \frac{1}{T}\int_0^T \left[\frac{R\sigma^2}{2}(a - \omega)(\partial_\theta \ln p)\left(1 - \frac{2\pi}{p\int_0^{2\pi} p^{-1}\,\mathrm{d}\theta}\right)\right]\mathrm{d}s.}_{=:I_3}
\end{aligned} \tag{19}$$

Next, Assumption 2 gives,

$$
\begin{aligned}
I_1 &= \int_0^{2\pi} p(\theta;\omega) \cdot \left[\bar{c}(\theta) + \frac{R\sigma^4}{8} (\partial_\theta \ln p)^2(\theta;\omega) \right] \mathrm{d}\theta, \\
I_2 &= \int_0^{2\pi} p(\theta;\omega) \cdot \left[\frac{R}{2} (\omega - a)^2 \left(1 - \frac{2\pi}{p \int_0^{2\pi} p^{-1} \, \mathrm{d}\theta} \right)^2 \right] \mathrm{d}\theta, \\
I_3 &= \int_0^{2\pi} p(\theta;\omega) \cdot \left[\frac{R\sigma^2}{2} (a - \omega)(\partial_\theta \ln p) \left(1 - \frac{2\pi}{p \int_0^{2\pi} p^{-1} \, \mathrm{d}\theta} \right) \right] \mathrm{d}\theta, \\
&= \int_0^{2\pi} \frac{R\sigma^2}{2} (a - \omega) \partial_\theta p \, \mathrm{d}\theta - \int_0^{2\pi} \frac{R\sigma^2}{2} (a - \omega) \frac{2\pi}{\int_0^{2\pi} p^{-1} \, \mathrm{d}\theta} (\partial_\theta \ln p) \, \mathrm{d}\theta = 0.
\end{aligned}
$$

Substituting I_1, I_2 and I_3 back in (19), we obtain the following expression for η:

$$
\eta(p; \bar{c}, \omega, a) = \int_0^{2\pi} p \left[\bar{c} + \frac{R\sigma^4}{8} (\partial_\theta \ln p)^2 + \frac{R}{2} (\omega - a)^2 \left(1 - \frac{2\pi}{p \int_0^{2\pi} p^{-1} \, \mathrm{d}\theta} \right)^2 \right] \mathrm{d}\theta. \tag{20}
$$

Through a substitution of the form $v^2(\theta;\omega) = p(\theta;\omega)$, we arrive at an expression for $\eta^g(v; \bar{c}, \omega, a)$ in (13). Since $p = v^2$ and p is a density function, we obtain the constraint (14). Finally, the constraint (17) is the consistency requirement of the mean field approximation. $\square$

Next we clarify the relationship between a solution of $\mathrm{V}_\infty^{\mathcal{G}}$ and a MFE of $\mathcal{G}_\infty$ and demonstrate how one may be constructed from the other. In [31], we have shown that the mean field PDE model (11a)-(11c) is also an infinite-population limit model of the game-theoretic problem. This suggests that there could be a certain relationship between the variational formulation and the PDEs. The following Proposition formalizes this relationship and shows that a solution to (15)-(17) solves the nonlinear eigenvalue problem of the PDEs (11a)-(11c). In fact, one may proceed in the reverse direction and derive a MFE of $\mathcal{G}_\infty$, given a solution of $\mathrm{V}_\infty^{\mathcal{G}}$.

Proposition 1. *The following hold:*

(i) Suppose (h, p, η^*), a MFE of $\mathcal{G}_\infty$, be a traveling wave solution of the PDEs (11a)-(11c) *with wave-speed a. Let $v = \sqrt{p}$. Then (v, η^*) is the solution of the nonlinear eigenvalue problem* (15)-(17).

(ii) Conversely, suppose (v, η^*) is a solution of the nonlinear eigenvalue problem (15)-(17). *Let $p(\theta, t; \omega) = v^2(\theta - at; \omega)$, and suppose $h(\theta, t; \omega) = h(\theta - at, 0; \omega)$ satisfies*

$$
\partial_\theta h = -\frac{R\sigma^2}{2} \partial_\theta \ln p - R(a - \omega) \left(1 - \frac{2\pi}{p \int_0^{2\pi} p^{-1} \, \mathrm{d}\theta} \right), \tag{21}
$$

where a is traveling wave speed. Then (h, p, η^*) is the MFE of $\mathcal{G}_\infty$.

In summary, solutions of $\mathcal{G}_\infty$ can be obtained by considering one of two problems:

1. The variational problem (13)-(14) with constraint (17);
2. The nonlinear eigenvalue problem (15)-(17).

3.3 Variational formulation of $\mathcal{W}_\infty$

As with the game-theoretic problem, we now introduce a variational problem, $\mathrm{V}_\infty^{\mathcal{W}}$, that serves as counterpart of the welfare optimization problem $\mathcal{W}_\infty$, associated with (4). This problem requires a $v \in \mathbf{X}$ that minimizes

$$\min_{v\in\mathbf{X}} \quad \eta^{\mathrm{w}}(v;a) := \int_\Omega \int_0^{2\pi} \Big[\mathcal{C}[v]v^2 + \frac{R\sigma^4}{2}(\partial_\theta v)^2 + \frac{R}{2}(\omega-a)^2 v^2 \left(1 - \frac{2\pi}{v^2 \int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\right)^2 \Big] \,\mathrm{d}\theta g(\omega)\,\mathrm{d}\omega \tag{22}$$

$$\text{s.t.} \quad 1 = \int_0^{2\pi} v^2(\theta;\omega)\,\mathrm{d}\theta, \tag{23}$$

where $\mathcal{C}[v]$ is defined as in (17). If we denote the average cost obtained through this avenue as $\eta_{\mathrm{w}}^*(a)$ i.e., $\eta_{\mathrm{w}}^*(a) := \min_{v\in\mathbf{X}} \eta^{\mathrm{w}}(v;a)$, then the Euler-Lagrange conditions may be derived by the next Lemma. Note that this Lemma combines Lemma 2 and Lemma 3 from the previous section.

Lemma 4. *Suppose $v^*(\theta;\omega)$ is a solution of ($V_\infty^{\mathcal{W}}$), given by (22)-(23). Then $(v^*(\theta;\omega), \lambda^*(\omega,a))$ satisfy the following:*

$$\partial_{\theta\theta}^2 v + \frac{2}{\sigma^4 R}(\lambda - 2\mathcal{C}[v])v - \frac{(\omega-a)^2}{\sigma^4}\left(1 - \left(\frac{2\pi}{v^2\int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\right)^2\right) v = 0, \tag{24}$$

$$\int_0^{2\pi} v^2(\theta;\omega)\,\mathrm{d}\theta - 1 = 0, \tag{25}$$

where λ is the Lagrange multiplier associated with (23). Furthermore,

$$\eta_w^*(a) = \int_\Omega \left(\lambda^*(\omega,a) - \int_0^{2\pi} \mathcal{C}[v^*](\theta)(v^*)^2(\theta;\omega)\,\mathrm{d}\theta\right) g(\omega)\,\mathrm{d}\omega. \tag{26}$$

In what follows, we present the main result of this subsection. Here, we show that a mean-field optimum of $\mathcal{W}_\infty$ is indeed a minimizer of the variational problem $V_\infty^{\mathcal{W}}$, a result that is analogous to Theorem 2.

Theorem 3. *Consider an Mean-Field Optimum (MFO) of $\mathcal{W}_\infty$. Under Assumption 2, any MFO of $\mathcal{W}_\infty$ is a minimizer of the variational problem $V_\infty^{\mathcal{W}}$ given by (22)-(23).*

Proof. We begin by recalling the welfare optimization problem

$$\eta^{(\mathrm{OPT})}(u) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^N \left[c(\theta_i;\theta_{-i}) + \tfrac{1}{2}Ru_i^2\right]\,\mathrm{d}s. \tag{27}$$

The mean-field approximation together with the traveling wave assumption (see Assumption 2) are used to simplify the two terms in the integrand.

(i) In the infinite-N limit,

$$\frac{1}{N}\sum_{i=1}^{N} c(\theta_i;\theta_{-i}) \approx \int_\Omega\int_0^{2\pi} p(\theta-as;\omega')\int_\Omega\int_0^{2\pi} c^\bullet(\theta-\vartheta)p(\vartheta-as;\omega)g(\omega)\,\mathrm{d}\vartheta\,\mathrm{d}\omega g(\omega')\,\mathrm{d}\theta\,\mathrm{d}\omega'$$

(ii) Likewise, using Lemma 1,

$$u = \frac{\sigma^2}{2}\partial_\theta \ln p + \left(1 - \frac{2\pi}{p\int_0^{2\pi}\frac{1}{p}\,\mathrm{d}\theta}\right)(a-\omega).$$

Substituting these in (27),we obtain the approximation of $\eta^{(\mathrm{OPT})}(u)$ as

$$\begin{aligned}
\eta^{\mathrm{W}}(p;a) = &\lim_{T\to\infty}\frac{1}{T}\int_0^T \Bigg(\int_\Omega\int_0^{2\pi} p(\theta-as;\omega')\int_\Omega\int_0^{2\pi} c^\bullet(\theta-\vartheta)p(\vartheta-as;\omega)g(\omega)\,\mathrm{d}\vartheta\,\mathrm{d}\omega \\
&\qquad g(\omega')\,\mathrm{d}\theta\,\mathrm{d}\omega'\Bigg)\,\mathrm{d}s \\
&+\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega\int_0^{2\pi} p(\theta-as;\omega)\tfrac{1}{2}Ru^2 g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\theta\,\mathrm{d}s \\
=:& I_1 + I_2.
\end{aligned} \tag{28}$$

This is simplified as follows:

$$\begin{aligned}
I_1 &= \int_\Omega\int_0^{2\pi} p(\theta;\omega')\int_\Omega\int_0^{2\pi} c^\bullet(\theta-\vartheta)p(\vartheta;\omega)\,\mathrm{d}\vartheta g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\theta g(\omega')\,\mathrm{d}\omega' \\
&= \int_\Omega\int_0^{2\pi} v^2(\theta;\omega')\mathcal{C}[v](\theta)\,\mathrm{d}\theta g(\omega')\,\mathrm{d}\omega',
\end{aligned} \tag{29}$$

where the equation is obtained from change of variable ($\theta - as$ to θ) and last equality is obtained from $p = v^2$ and the definition of $\mathcal{C}[v]$.

$$\begin{aligned}
I_2 &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega\int_0^{2\pi} p(\theta-as;\omega)\tfrac{1}{2}Ru^2 g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\theta\,\mathrm{d}s \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega\int_0^{2\pi} p(\theta;\omega)\tfrac{1}{2}R\left(\frac{\sigma^2}{2}\partial_\theta\ln p + \left(1-\frac{2\pi}{p\int p^{-1}}\right)(a-\omega)\right)^2 g(\omega)\,\mathrm{d}\theta\,\mathrm{d}\omega\,\mathrm{d}s \\
&= \int_\Omega\int_0^{2\pi} p(\theta;\omega)\tfrac{1}{2}R\left(\frac{\sigma^2}{2}\partial_\theta\ln p + \left(1-\frac{2\pi}{p\int p^{-1}}\right)(a-\omega)\right)^2 g(\omega)\,\mathrm{d}\theta\,\mathrm{d}\omega \\
&= \int_\Omega\int_0^{2\pi}\Bigg[\frac{R\sigma^4}{8}\frac{(\partial_\theta p)^2}{p} + \frac{Rp}{2}(\omega-a)^2\left(1-\frac{2\pi}{p\int p^{-1}}\right)^2 \\
&\qquad + \frac{R\sigma^2}{2}(a-\omega)\left(\partial_\theta p - \frac{2\pi}{\int p^{-1}}\partial_\theta \ln p\right)\Bigg] g(\omega)\,\mathrm{d}\theta\,\mathrm{d}\omega \\
&=: I_{21} + I_{22} + I_{23}
\end{aligned}$$

The 2π-periodic property of p gives $I_{23} = 0$. Letting $p = v^2$, we have

$$\begin{aligned} I_2 &= I_{21} + I_{22} \\ &= \int_\Omega \int_0^{2\pi} \left[\frac{R\sigma^4}{2} (\partial_\theta v)^2 + \frac{R}{2} v^2 (\omega - a)^2 \left(1 - \frac{2\pi}{v^2 \int v^{-2}} \right)^2 \right] g(\omega) \, \mathrm{d}\theta \, \mathrm{d}\omega. \end{aligned} \tag{30}$$

Substituting (29) and (30) into (28), we arrive at the expression for $\eta^{\mathrm{w}}(v; a)$ in (22). Since $p = v^2$ and p is a density function, we obtain the constraint (23). The required result follows. $\square$

In summary, solutions of $\mathcal{W}_\infty$ can be obtained by considering the variational problem ($\mathrm{V}_\infty^{\mathcal{W}}$) given by (22)-(23). Note that the nonlinear eigenvalue problem (24)-(25) represents the necessary conditions for solutions of $\mathcal{W}_\infty$ and currently a stronger result, as captured by Prop. 1 in the context of mean-field equilibria, is unavailable. As a consequence, we now have access to MFE of $\mathcal{G}_\infty$ and MFO of $\mathcal{W}_\infty$ through the solution of suitably defined variational problems, denoted by $\mathrm{V}_\infty^{\mathcal{G}}$ and $\mathrm{V}_\infty^{\mathcal{W}}$ respectively.

4 Efficiency loss

This section focuses on the analysis of efficiency loss. The efficiency loss characterization is carried out with respect to the solutions of the Euler-Lagrange BVP, (15)-(17) for the game and (24)-(25) for the welfare problem.

In Section 4.1, we provide a relationship between a MFE and an MFO which allows for constructing an expression for the efficiency loss. Then, in Section 4.2, we utilize bifurcation analysis to derive bounds using the aforementioned expression for the efficiency loss.

There are several insights that one can draw from the expression for $\Delta_\eta(R)$, particularly from the numerical study. We observe that as $R \to 0$, we have that η_{w}^* and η_g^* both tend to zero. In effect, the efficiency loss tends to zero, as $R \to 0$. Furthermore, when R is beyond a threshold, we again observe that the efficiency loss is zero. In fact, the efficiency loss is seen to be positive between these two regimes. In the following, through a bifurcation analysis, we provide a locally valid upper bound on efficiency loss (Lemma 2) for homogeneous case, i.e., $\gamma = 0$ and thus $\omega_i = 1$ for all $i = 1, \dots, N$. Then we provide the numerical results in the next section.

4.1 Relating the MFO and MFE

In this subsection, we examine the efficiency loss associated with using a MFE. This loss is denoted by $\Delta_\eta(R; \sigma, a)$ which is defined as follows:

$$\Delta_\eta(R; \sigma, a) \triangleq \mathrm{E}_\omega \left[\eta_g^*(R; \sigma, \omega, a) \right] - \eta_{\mathrm{w}}^*(R; \sigma, a), \tag{31}$$

where $\mathrm{E}_\omega[\cdot] := \int_\Omega \cdot g(\omega) \, \mathrm{d}\omega$. We provide two sets of results in this subsection. First, we provide a precise relationship between a MFE and an MFO, in terms of v and λ. Second, using these relationships, we construct an expression for $\Delta_\eta(R)$.

Theorem 4. *Let σ and wave speed a be fixed. For a given value of R, let $(\bar{c}^*, v_g^*(R))$ be the solution of $V_\infty^{\mathcal{G}}$ and $\lambda_g^*(R)$ be the corresponding Lagrange multiplier. Suppose $v_w^*(R)$ denotes the solution to the welfare variational problem $V_\infty^{\mathcal{W}}$ and $\lambda_w^*(R)$ denotes the corresponding Lagrange multiplier. Then we have the following:*

(i) $v_w^*(R) = v_g^*(R/2)$, $\lambda_w^*(R) = 2\lambda_g^*(R/2)$,

(ii) $\Delta_\eta(R) = E_\omega\Big[\lambda_g^*(R) - 2\lambda_g^*(\frac{R}{2}) + \int_0^{2\pi}\!\!\int_0^{2\pi} c^\bullet(\theta-\vartheta)(v_g^*)^2(\vartheta; \frac{R}{2})\,\mathrm{d}\vartheta (v_g^*)^2(\theta; \frac{R}{2})\,\mathrm{d}\theta\Big]$.

Proof. (i) Denote the equation (15) as $G^g(v,\lambda,\omega,R) = 0$ and the equation (24) as $G^{\mathrm{w}}(v,\lambda,\omega,R) = 0$. Consider the problem $G^{\mathrm{w}}(v^{\mathrm{w}},\lambda^{\mathrm{w}},\omega,R^{\mathrm{w}}) = 0$. Suppose $R^{\mathrm{w}} = 2R$ and $\lambda^{\mathrm{w}} = 2\lambda$. Then we obtain the relationship

$$\begin{aligned} G^{\mathrm{w}}(v^{\mathrm{w}},\lambda^{\mathrm{w}},\omega,R^{\mathrm{w}}) &= \partial^2_{\theta\theta} v^{\mathrm{w}} + \frac{2}{\sigma^4 2R}(2\lambda - 2\mathcal{C}[v^{\mathrm{w}}](\theta))v^{\mathrm{w}} \\ &= \partial^2_{\theta\theta} v^{\mathrm{w}} + \frac{2}{\sigma^4 R}(\lambda - \mathcal{C}(v^{\mathrm{w}})(\theta))v^{\mathrm{w}} \\ &= G^g(v^{\mathrm{w}},\lambda,\omega,R) = G^g(v^{\mathrm{w}},\lambda^{\mathrm{w}}/2,\omega,R^{\mathrm{w}}/2). \end{aligned}$$

That is to say, to solve the problem $G^{\mathrm{w}}(v^{\mathrm{w}},\lambda^{\mathrm{w}},R) = 0$, we could instead solve the equivalent problem $G^g(v^g,\lambda^g,R/2) = 0$. Then $v^{\mathrm{w}}(R) = v^g(R/2)$ and $\lambda^{\mathrm{w}}(R) = 2\lambda^g(R/2)$.

(ii) An expression for Δ_η may be obtained from its definition (31) using the relationship in (i) and Eqn. (26).

□

We are now in a position to examine the loss of welfare associated with a MFE and this represents the focus of the following.

4.2 Local bound through bifurcation analysis

In this section, we investigate the solutions of the nonlinear eigenvalue problems (15)-(17) and (24)-(25) by using bifurcation analysis and conclude with a local bound on the efficiency loss. Details on bifurcation theory may be found in the monography by Iooss and Joseph [12] and an expansive description of bifurcation analysis in the context of mean-field oscillator games is provided in [32]. We only consider the homogeneous case, i.e., $a = \omega_i = 1$ for all $i = 1,\ldots,N$.

We denote $\mathbf{Y} := C^0_{2\pi}([0,2\pi],\mathbb{R})$, the space of continuous real-valued periodic functions on $[0,2\pi]$. Recall $\mathbf{X} := C^2_{2\pi}([0,2\pi],\mathbb{R}^+)$. The eigenvalue problem (Denoted as (EP$^\alpha$)) comprises of a nonlinear operator $G_\alpha : \mathbf{X}\times\mathbb{R}^+\times\mathbb{R}^+ \to \mathbf{Y}$, and a constraint $B : \mathbf{X} \to \mathbb{R}$:

$$G_\alpha(v,\lambda,R) := \partial^2_{\theta\theta} v + \frac{2}{\sigma^4 R}\left(\lambda - \alpha\mathcal{C}(v)\right)v = 0, \tag{32}$$

$$B(v) := \int v^2(\theta)\,\mathrm{d}\theta - 1 = 0, \tag{33}$$

where $\mathcal{C}(v)$ is defined in (17) and $\alpha = 1, 2$. Note that $\alpha = 1$ refers to the nonlinear eigenvalue problem associated with the game-theoretic problem, as specified by (15)-(17), while $\alpha = 2$

refers to the corresponding problem arising from the welfare problem, as given by (24)-(25). Given a fixed $R \in \mathbb{R}^+$, we are interested in obtaining *solutions* $(v, \lambda) \in \mathbf{X} \times \mathbb{R}^+$ that satisfy $G_\alpha(v, \lambda, R) = 0$ and $B(v) = 0$, for $\alpha = 1, 2$.

In the context of the nonlinear eigenvalue problem, we define the incoherence solution

$$
\begin{aligned}
v &= v_0 := \frac{1}{\sqrt{2\pi}}, \\
\lambda &= \lambda_0 := \alpha C_0^\bullet = \frac{\alpha}{2\pi} \int_0^{2\pi} c^\bullet(\theta) \, \mathrm{d}\theta.
\end{aligned}
$$

About the incoherence solution, the linearization of (32) is given by

$$
\mathcal{L}_R \tilde{v}(\theta) := \partial^2_{\theta\theta} \tilde{v} - \frac{2\alpha}{\sigma^4 R \pi} \int_0^{2\pi} c^\bullet(\theta - \vartheta) \tilde{v}(\vartheta) \, \mathrm{d}\vartheta
$$

with $\tilde{v} \in \mathbf{X}$ and satisfies the integral constraint $\int_0^{2\pi} \tilde{v}(\theta) \, \mathrm{d}\theta = 0$.

The spectrum of the linear operator $\mathcal{L}_R : \mathbf{X} \to \mathbf{Y}$ is summarized in the following:

Theorem 5. *Consider the linear eigenvalue problem $\mathcal{L}_R v = sv$. Suppose the Fourier expansion of the function $c^\bullet$ is*

$$
c^\bullet(\theta) = \sum_{k=-\infty}^{\infty} C_k^\bullet e^{ik\theta}. \tag{34}
$$

Then the spectrum consists of eigenvalues $s = -k^2 - \frac{4\alpha}{\sigma^4 R} C_k^\bullet =: s_k$ for $k = 0, 1, 2, \ldots$ The eigenspace for the k^{th} eigenvalue $s = s_k$ is given by span$\{\cos(k\theta), \sin(k\theta)\}$. ∎

As the parameter R varies, the potential bifurcation points are where an eigenvalue crosses zero. The k^{th} such bifurcation point is given by

$$
R = R_k := -\frac{4\alpha}{k^2 \sigma^4} C_k^\bullet.
$$

Example 1. *Consider now the function $c^\bullet(\theta - \vartheta) = \frac{1}{2} \sin^2\left(\frac{\vartheta - \theta}{2}\right)$. In this case, $C_1^\bullet = -\frac{1}{8}$ and the first bifurcation point, defined as*

$$
R = \frac{\alpha}{2\sigma^4} =: R_c^\alpha, \tag{35}
$$

is the critical point at which the incoherence solution loses stability.

A Lyapunov-Schmidt perturbation method is used to obtain a (formal) asymptotic formula for the non-constant bifurcating solution branch. We substitute the expansion

$$
\begin{aligned}
R &= r_0 + x r_1 + x^2 r_2 + \ldots \\
v &= v_0 + x v_1 + x^2 v_2 + \ldots \\
\lambda &= \lambda_0 + x \lambda_1 + x^2 \lambda_2 + \ldots
\end{aligned} \tag{36}
$$

into (32)-(33), and collect the terms according to different orders of x. The results are summarized in the following lemma with the calculations provided in the Appendix 7.6.

Lemma 5. *Given the function* $c^{\bullet}(\vartheta-\theta)=\frac{1}{2}\sin^2\left(\frac{\vartheta-\theta}{2}\right)$*, the synchrony solution for* (32)-(33) *is given by an asymptotic formula in the small "amplitude" parameter* x *when* $R<r_0$*:*

$$\begin{aligned}
v(x) =& v_0 + 2x\cos(\theta+\theta_0) + \left(-\frac{1}{v_0} + v_0\pi\cos 2(\theta+\theta_0)\right)x^2 + O(x^3),\\
\lambda =& \hat{\lambda}(x) = \lambda_0 - \alpha\pi x^2 + O(x^3),\\
R =& \hat{R}(x) = r_0 - \frac{7\alpha\pi}{2\sigma^4}x^2 + O(x^3),
\end{aligned} \tag{37}$$

where $r_0 = R_c^{\alpha} = \frac{\alpha}{2\sigma^4}$, $\lambda_0 = \frac{\alpha}{4}$ *for* $\alpha=1,2$ *and* θ_0 *is an arbitrary phase in* $[0,2\pi]$.

Using the asymptotic formula from lemma 5 and the formula of $\Delta_\eta(R)$ in Thm. 4, we obtain an upper bound for the efficiency loss around the critical value of R_c^1 and is provided in the following.

Proposition 2. *In a sufficiently small neighborhood of* $R=R_c^1=\frac{1}{2\sigma^4}$*, the following bound holds for* Δ_η*:*

$$\Delta_\eta(R) \le \frac{6}{49}(1-\sigma^4 R)^2 + O(x^3(1,R/2)), \tag{38}$$

where $x(\alpha,R)=\sqrt{\frac{2\sigma^4}{7\alpha\pi}(R_c^{\alpha}-R)}$ *and* R_c^{α} *is defined as* (35)*.*

Proof. We prove the results using the approximation (37) and Thm. 4 (ii). Denote the third term of $\Delta_\eta(R)$ in Thm. 4 (ii) as Δ_η^p. Then we have

$$\Delta_\eta(R) = \lambda_g^*(R) - 2\lambda_g^*(\frac{R}{2}) + \Delta_\eta^p,$$

where $\Delta_\eta^p = \int\int c^{\bullet}(\theta-\vartheta)[v_g^*(\vartheta;\frac{R}{2})]^2\,\mathrm{d}\vartheta[v_g^*(\theta;\frac{R}{2})]^2\,\mathrm{d}\theta$. The bound is provided around the point R_c^1. So we consider the two situations: 1) $R\le R_c^1$; 2) $R_c^1 < R (\le R_c^2)$.

First, we consider $R\le R_c^1$. We have from $(37)_3$

$$x^2(\alpha,R) = \frac{2\sigma^4}{7\alpha\pi}(R_c^{\alpha}-R) \quad \forall R\le R_c^{\alpha}. \tag{39}$$

Substitute x^2 into $(37)_2$, we obtain

$$\lambda_g^*(R) = \lambda_0^{\alpha} - \alpha\pi\times\frac{2\sigma^4}{7\alpha\pi}(R_c^{\alpha}-R) + O(x^3)\Big|_{\alpha=1} = \frac{1}{4} - \frac{2\sigma^4}{7}(R_c^1-R) + O(x^3), \tag{40}$$

and

$$\begin{aligned}
\Delta_\eta(R) =& \Delta_\eta^p + \left(\frac{1}{4} - \frac{2\sigma^4}{7}(R_c^1-R)\right) - 2\left(\frac{1}{4} - \frac{2\sigma^4}{7}(R_c^1-\frac{R}{2})\right)\\
&+ O(\max\{x^3(1,R), x^3(1,R/2\})\\
=& \Delta_\eta^p + \frac{1}{7} - \frac{1}{4} + O(x^3(1,R/2)).
\end{aligned} \tag{41}$$

We estimate Δ^p_η using the approximation of v in (37)$_1$.

$$
\begin{aligned}
\left(v_g^*(\theta;R)\right)^2 &= v_0^2 + 4x^2\cos^2(\theta+\theta_0) + 4v_0 x\cos(\theta+\theta_0) + 2v_0\left(-\frac{1}{v_0} + v_0\pi\cos 2(\theta+\theta_0)\right)x^2 \\
&\quad + 4\cos(\theta+\theta_0)\left(-\frac{1}{v_0} + v_0\pi\cos 2(\theta+\theta_0)\right)x^3 + O(x^3)\Big|_{x=x(1,R)} \\
&= v_0^2 + x4v_0\cos(\theta+\theta_0) + x^2 3\cos 2(\theta+\theta_0) + x^3\left(2v_0\pi - \frac{4}{v_0}\right)\cos(\theta+\theta_0) \\
&\quad + O(x^3)\Big|_{x=x(1,R)}.
\end{aligned} \tag{42}
$$

Substituting $c^\bullet(\theta-\vartheta) = \frac{1}{2}\sin^2\left(\frac{\theta-\vartheta}{2}\right)$ and (42) into Δ^p_η, we obtain

$$
\begin{aligned}
\Delta^p_\eta &= \frac{1}{4}\left(1 - \int_0^{2\pi}\int_0^{2\pi}\cos(\theta-\vartheta)\left(v_g^*(\vartheta;R/2)\right)^2\,\mathrm{d}\vartheta\left(v_g^*(\theta;R/2)\right)^2\,\mathrm{d}\theta\right) \\
&= \frac{1}{4} - \frac{1}{4}\left(\int_0^{2\pi}\cos(\theta+\theta_0)\left(v_g^*(\vartheta;R/2)\right)^2\,\mathrm{d}\theta\right)^2 \\
&= \frac{1}{4} - \frac{1}{4}\left(4\pi v_0 x - 3\pi\sqrt{2\pi}x^3\right)^2 + O(x^6)\Big|_{x=x(1,R/2)} \\
&\le \frac{1}{4} - 2\pi x^2 + 6\pi^2 x^4 + O(x^6)\Big|_{x=x(1,R/2)} \\
&= \frac{1}{4} - \frac{2}{7}(1-\sigma^4 R) + \frac{6}{49}(1-\sigma^4 R)^2 + O(x^6(1,R/2)) \quad \text{by (39)},
\end{aligned} \tag{43}
$$

and together with (41), we have

$$
\begin{aligned}
\Delta_\eta(R) &\le \frac{1}{7} - \frac{2}{7}(1-\sigma^4 R) + \frac{6}{49}(1-\sigma^4 R)^2 + O(\max\{x^6(1,R/2), x^3(1,R/2)\}) \\
&\le \frac{6}{49}(1-\sigma^4 R)^2 + O(x^3(1,R/2)), \ \text{ since } R \le R_c^1.
\end{aligned} \tag{44}
$$

Next, we consider $R > R_c^1$. In this case, the game is in incoherence. So $\lambda_g^*(R) \equiv \frac{1}{4}$. Since $R \le R_c^2$, we have $\lambda_g^*(R/2)$ same as in (40) and Δ^p_η same as in (43). Therefore, we get

$$
\begin{aligned}
\Delta_\eta(R) &= \frac{1}{4} - 2\left(\frac{1}{4} - \frac{2\sigma^4}{7}(R_c^1 - \frac{R}{2})\right) + \Delta^p_\eta + O(x^3(1,R/2)) \\
&= -\frac{1}{4} + \frac{2}{7}(1-\sigma^4 R) + \Delta^p_\eta + O(x^3(1,R/2)) \\
&\le \frac{6}{49}(1-\sigma^4 R)^2 + O(x^3(1,R/2)).
\end{aligned} \tag{45}
$$

The result (38) is proved by combining (44) and (45). □

5 Numerics

In this section we present the numerical results of the nonlinear eigenvalue problem. For the homogeneous case, we also compare the solution obtained using the perturbation method

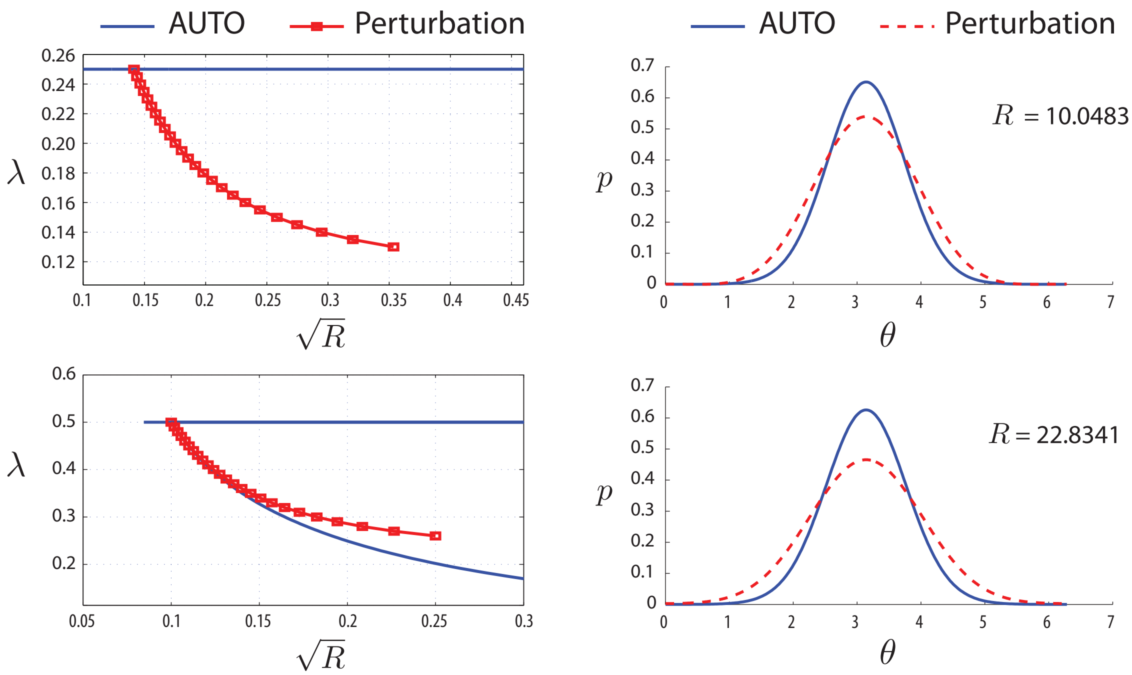

Figure 1: (Left) Bifurcation diagram for the Lagrange multiplier λ as a function of parameter $1/\sqrt{R}$; First row for (EP^1) of ($\mathrm{V}_\infty^{\mathcal{G}}$) and second row for (EP^2) of ($\mathrm{V}_\infty^{\mathcal{W}}$). For (EP^1), λ also equals the average cost η_g^*. (Right) The solution $v^2(\theta)$; $R = 10$ ($R^{-1/2} = 0.31$) for (EP^1) and $R = 22.8$ ($R^{-1/2} = 0.21$) for (EP^2).

against a numerical solution. The numerical solution of the eigenvalue problem is obtained by using a numerical continuation software AUTO [7]. For the heterogeneous case, the numerical results are presented.

5.1 Homogeneous case

We first consider the homogeneous nonlinear eigenvalue problem (32)-(33) for $c^{\bullet}(\theta-\vartheta)=\frac{1}{2}\sin^2\left(\frac{\theta-\vartheta}{2}\right)$. We set the noise level at $\sigma^2=0.1$. The results of the solutions p and the average cost η^* from Lyapunov-Schmidt perturbation method as well as those from the AUTO software are depicted for comparison.

Relationship of p and λ with R Figure 1 depicts the bifurcation diagram for the Lagrange multiplier λ as a function of the bifurcation parameter R as well as the function of p for a particular value of R ($R=10$ for (EP^1) and $R=22.8$ for (EP^2)). For comparison, we also depict the numerical solution that is obtained by using AUTO [7]. The first row in the figure depicts the results for (EP^1) and the second row depicts the results for (EP^2). These comparisons also serve to verify the perturbation results of Sec. 4.2.

Relationship of η with R Next, we compare the average cost (η_g^* for (EP^1) and η_{w}^* for (EP^2)) obtained from solving the two nonlinear eigenvalue problems using AUTO. For (EP^1), we know $\eta_g^*=\lambda$ from lemma 3. For (EP^2), we know η_{w}^* from lemma 4. The results are depicted in Fig. 2 (Left). There are two critical points in the figure: One is R_c^1 for (EP^1) and the other is R_c^2 for (EP^2). When $R>R_c^2$ ($R^{-1/2}<(R_c^2)^{-1/2}$), we obtain the incoherence solution for both problems. When $R<R_c^1$ ($R^{-1/2}>(R_c^1)^{-1/2}$), we obtain the synchrony solution for both problems. The figure shows $\eta_g^*\geq\eta_{\mathrm{w}}^*$. The equality holds when both are in incoherence solution, i.e., $R>R_c^2$.

Relationship of Δ_η with R We calculate the difference of the average cost (efficiency loss) Δ_η for the case of $R<R_c^1$. The difference is calculated by two methods. One is the method stated in Thm. 4 (ii) and the other is the definition. The results are depicted in Fig. 2 (Right). It shows that the formula for $\Delta_\eta(R)$ in Thm. 4 is quite accurate, and the solution of ($\mathrm{V}_\infty^{\mathcal{G}}$) is always inefficient except in the incoherent regime. From the numerics, we obtain that $\Delta_\eta/\eta_{\mathrm{w}}^*<20\%$, while we get $\Delta_\eta\leq 0.03$ from (38), which gives $\Delta_\eta/\eta_{\mathrm{w}}^*<15\%$.

5.2 Heterogeneous case

In this section, we depict the numerical results for the heterogeneous nonlinear eigenvalue problem:

$$\begin{aligned}G_\alpha(v,\lambda,R,\omega,a):=\partial_{\theta\theta}^2 v+\frac{2}{\sigma^4 R}\left(\lambda-\alpha\mathcal{C}(v)\right)v\\
-\frac{(\omega-a)^2}{\sigma^4}\left(1-\left(\frac{2\pi}{v^2\int_0^{2\pi}v^{-2}\,\mathrm{d}\theta}\right)^2\right)v=0,\end{aligned}\tag{46}$$

$$B(v):=\int v^2(\theta)\,\mathrm{d}\theta-1=0,\tag{47}$$

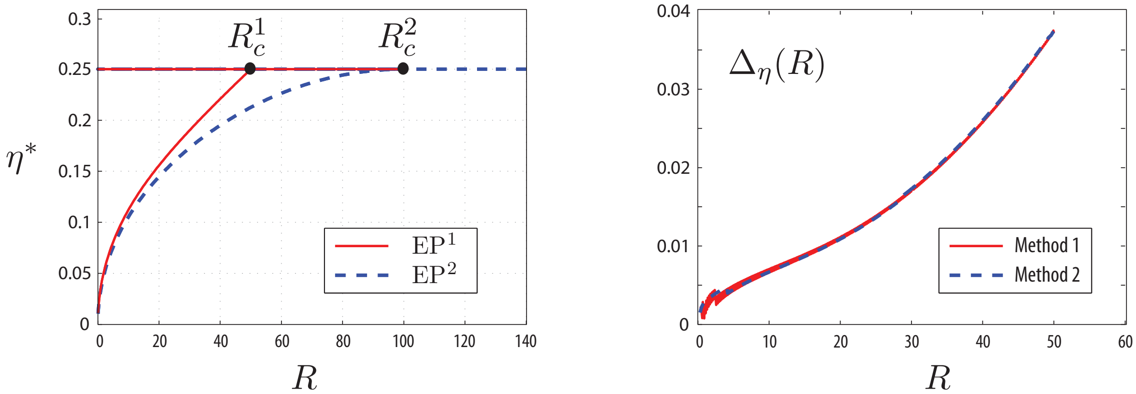

Figure 2: (Left) The bifurcation diagram in terms of the average cost η^* for two eigenvalue problems (EP$^\alpha$). The results are for $\sigma^2 = 0.1$. The critical value of R for (EP^1) of the game problem is $R_c^1 = 50\,((R_c^1)^{-1/2} = 0.1414)$ while for (EP^2) of the welfare optimization problem is $R_c^2 = 100\,((R_c^2)^{-1/2} = 0.1)$. (Right) Δ_η calculated using two methods: Method one is the method stated in Thm. 4 (ii); Method two is the definition $\Delta_\eta(R) = \eta_g^*(R) - \eta_{\mathrm{w}}^*(R)$.

where $\mathcal{C}(v)$ is defined in (17) and $\alpha = 1, 2$. When $\alpha = 1$, it is the problem (15)-(17), while when $\alpha = 2$, it is the problem (24)-(25). These results are obtained numerically using the AUTO software.

For the numerical computation, we sample three uniformly distributed frequencies: $\omega_1 = 0.95$, $\omega_2 = 1.00$ and $\omega_3 = 1.05$. The traveling wave speed is set at the mean of frequencies, i.e., $a = \mathrm{E}[\omega] = 1$.

Relationship of η with R We compare the average cost ($\mathrm{E}_\omega[\eta_g^*]$ for (EP^1) and η_{w}^* for (EP^2)) obtained from solving the two nonlinear eigenvalue problems using AUTO. For (EP^1), we know $\eta_g^*(\omega, a) = \lambda^*(\omega, a)$ from lemma 3. For (EP^2), we know η_{w}^* from lemma 4. The results are depicted in Fig. 3 (Left). There are two critical points in the figure: One is R_c^1 for (EP^1) and the other is R_c^2 for (EP^2). When $R > R_c^2$ ($R^{-1/2} < (R_c^2)^{-1/2}$), we obtain the incoherence solution for both problems. When $R < R_c^1$ ($R^{-1/2} > (R_c^1)^{-1/2}$), we obtain the synchrony solution for both problems. The figure shows $\mathrm{E}_\omega[\eta_g^*] \geq \eta_{\mathrm{w}}^*$. The equality holds when both are in incoherence solution, i.e., $R > R_c^2$.

Relationship of Δ_η with R We calculate the difference of the average cost (efficiency loss) Δ_η for the case of $R < R_c^1$. The difference is calculated by two methods. One is the method stated in Thm. 4 (ii) and the other is the definition. The results are depicted in Fig. 2 (Right). It shows that the formula for $\Delta_\eta(R)$ in Thm. 4 is quite accurate, and the solution of (V$_\infty^{\mathcal{G}}$) is always inefficient except in the incoherent regime. From the numerics, we obtain that $\Delta_\eta/\eta_{\mathrm{w}}^* < 21.5\%$.

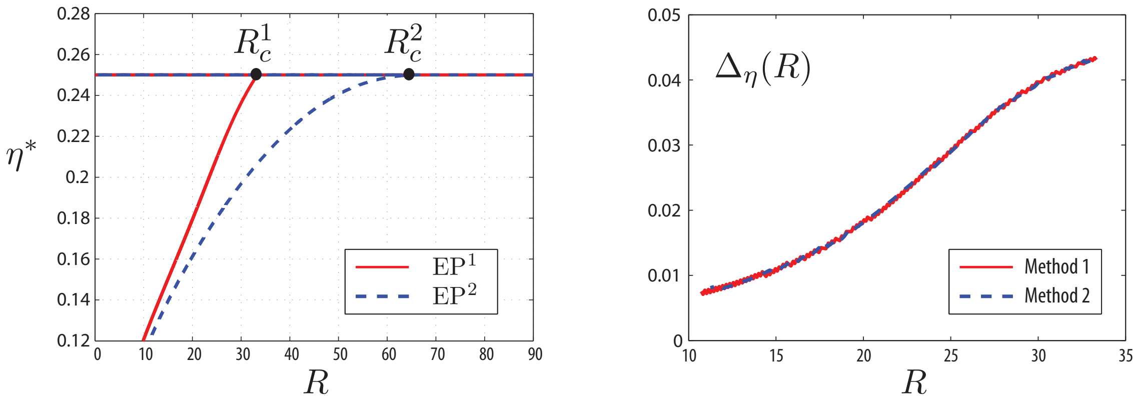

Figure 3: (Left) The bifurcation diagram in terms of the average cost $\mathrm{E}_\omega[\eta^*]$ for two eigenvalue problems (EP$^\alpha$). The results are for $\sigma^2 = 0.1$ and $a = 1$. The critical value of R for (EP^1) of the game problem is $R_c^1 = 33.33$ while for (EP^2) of the welfare optimization problem is $R_c^2 = 66.66$. (Right) Δ_η calculated using two methods: Method one is the method stated in Thm. 4 (ii); Method two is the definition $\Delta_\eta(R) = \mathrm{E}_\omega[\eta_g^*](R) - \eta_{\mathrm{w}}^*(R)$.

6 Conclusions

Quantification of efficiency loss is an important issue in the design of engineered multi-agent systems. While there has been prior work in static and deterministic regimes, in this paper, we provide results for a class of stochastic dynamic game-theoretic models. The analysis is based on the consideration of an idealized mean field limit model. Via variational techniques, nonlinear eigenvalue problems are introduced for the game-theoretic and welfare optimization problems, respectively. The two eigenvalue problems are shown to exhibit a simple relationship, which is then used to obtain a tractable expression for the efficiency loss. Local bounds on efficiency loss are derived by using bifurcation theory techniques in the homogeneous frequency case. The bounds are verified by using numerical techniques for the homogeneous case as well as the heterogeneous case.

Given the inherent challenge in developing efficiency estimates, our results are naturally characterized by some limitations. For instance, our analysis is restricted to a subclass of traveling wave solutions (as specified by Assumption 2). We also discuss conditions on the cost functions under which such solutions are known to exist (Assumption 1). While we cannot yet claim whether the existence of a MFE implies that the cooperative control problem admits a solution, our results show that within the regime of interest, solutions to both problems exist and can indeed be compared. This is a subject of continuing investigation.

7 Appendix

7.1 Proof of Lemma 1

Proof. Under Assumption 2, the Eqn. (11b) can be written as

$$(\omega - a)\partial_\theta p = -\partial_\theta[pu] + \frac{\sigma^2}{2}\partial^2_{\theta\theta}p.$$

Integrating both sides of the equation with respect to θ,

$$u = \frac{\sigma^2}{2}\frac{\partial_\theta p}{p} + (a-\omega) + \frac{K}{p}, \tag{48}$$

where K is a function of ω and is obtained as follows. Integrating both sides of the resulting equation (48) from 0 to 2π with respect to θ again, we obtain

$$\int_0^{2\pi} u \,\mathrm{d}\theta = \int_0^{2\pi} \frac{\sigma^2}{2}\partial_\theta \ln p \,\mathrm{d}\theta + K\int_0^{2\pi}\frac{1}{p}\,\mathrm{d}\theta + (a-\omega)2\pi.$$

From the assumption that h (thus u) and p are 2π-periodic in θ,

$$0 = 0 + K\int_0^{2\pi}\frac{1}{p}\,\mathrm{d}\theta + (a-\omega)2\pi \quad \Rightarrow \quad K = \frac{(\omega-a)2\pi}{\int_0^{2\pi}\frac{1}{p}\,\mathrm{d}\theta}. \tag{49}$$

Finally, we get the result (7) by substituting K in (49) back to (48). □

7.2 Proof of Lemma 2

We consider the functional $I[v] = I_1[v] + I_2[v] + I_3[v]$ where

$$\begin{aligned}
I_1[v] &= \int_0^{2\pi} \bar{c}v^2 \,\mathrm{d}\theta, \\
I_2[v] &= \int_0^{2\pi} \left(\frac{R\sigma^4}{2}(\partial_\theta v)^2 - \lambda v^2\right)\mathrm{d}\theta, \\
I_3[v] &= \int_0^{2\pi} \left(\frac{R}{2}(\omega-a)^2 v^2 \left(1 - \frac{2\pi}{v^2\int v^{-2}}\right)^2\right)\mathrm{d}\theta,
\end{aligned}$$

and derive its first variation. For $I_1[v]$,

$$DI_1[v]\cdot v' = \int_0^{2\pi} 2\bar{c}v\cdot v' \,\mathrm{d}\theta. \tag{50}$$

For $I_2[v]$,

$$DI_2[v]\cdot v' = \int_0^{2\pi}\left(-R\sigma^4\partial^2_{\theta\theta}v\cdot v' - 2\lambda v\cdot v'\right)\mathrm{d}\theta. \tag{51}$$

A straightforward calculation gives,

$$
\begin{aligned}
DI_3[v]\cdot v' &= \lim_{\epsilon\to 0}\frac{I_3[v+\epsilon v']-I_3[v]}{\epsilon}\\
&= \int_0^{2\pi} R(\omega-a)^2\left(1-\left(\frac{2\pi}{v^2\int\frac{1}{v^2}}\right)^2\right) v\cdot v'\,\mathrm{d}\theta
\end{aligned}
\tag{52}
$$

Using (50)-(52), we have obtain the nonlinear problem (15). Finally, (16) is the same constraint as (14).

7.3 Proof of Lemma 3

Proof. The proof of the first half is same as Lemma 2. It remains to show that $\lambda^*(\omega,a)=\eta_g^*(\omega,a)$. Multiplying both sides of (15) with $\frac{R\sigma^4 v^*}{2}$ and integrating from 0 to 2π with respect to θ, we obtain

$$
\begin{aligned}
\int_0^{2\pi}\Bigg[&\frac{R\sigma^4}{2}v^*\partial^2_{\theta\theta}v^*+(\lambda^*-\bar{c}^*)(v^*)^2\\
&-\frac{R}{2}(\omega-a)^2\left(1-\left(\frac{2\pi}{(v^*)^2\int(v^*)^{-2}}\right)^2\right)(v^*)^2\Bigg]\mathrm{d}\theta=0.
\end{aligned}
$$

Because $\int_0^{2\pi}(v^*)^2\,\mathrm{d}\theta=1$,

$$
\begin{aligned}
\lambda^* &= \int_0^{2\pi}\left[-\frac{R\sigma^4}{2}v^*\partial^2_{\theta\theta}v^*+\bar{c}^*(v^*)^2+\frac{R}{2}(\omega-a)^2\left(1-\left(\frac{2\pi}{(v^*)^2\int(v^*)^{-2}}\right)^2\right)(v^*)^2\right]\mathrm{d}\theta\\
&= -\frac{R\sigma^4}{2}v^*\partial_\theta v^*\Big|_0^{2\pi}+\int_0^{2\pi}\frac{R\sigma^4}{2}(\partial_\theta v^*)^2\,\mathrm{d}\theta\\
&\quad+\int_0^{2\pi}\left[\bar{c}^*(v^*)^2+\frac{R}{2}(\omega-a)^2\left(1-\left(\frac{2\pi}{(v^*)^2\int(v^*)^{-2}}\right)^2\right)(v^*)^2\right]\mathrm{d}\theta\\
&= \int_0^{2\pi}\left[\frac{R\sigma^4}{2}(\partial_\theta v^*)^2+\bar{c}^*(v^*)^2+\frac{R}{2}(\omega-a)^2\left(1-\left(\frac{2\pi}{(v^*)^2\int(v^*)^{-2}}\right)^2\right)(v^*)^2\right]\mathrm{d}\theta\\
&= \int_0^{2\pi}\left[\frac{R\sigma^4}{2}(\partial_\theta v^*)^2+\bar{c}^*(v^*)^2+\frac{R}{2}(\omega-a)^2\left(1-\frac{2\pi}{(v^*)^2\int(v^*)^{-2}}\right)^2(v^*)^2\right]\mathrm{d}\theta
\end{aligned}
\tag{53}
$$

where the second equality is obtained through integration by parts of the first term, and the third equality is obtained because v^* is periodic function with period 2π. From definition (13), the right hand side of (53) is $\eta_g^*(\omega,a)$. □

7.4 Proof of Proposition 1

Proof. (i) Let (h,p,η^*) be a solution to the PDE (11a)-(11c). Then the optimal control is given by $u^*=-\frac{1}{R}\partial_\theta h$. We have shown that u^* also satisfies (7). Therefore, we obtain

$$
\partial_\theta h=-\frac{R\sigma^2}{2}\partial_\theta\ln p-R(a-\omega)\left(1-\frac{2\pi}{p\int p^{-1}}\right).
\tag{54}
$$

Taking partial derivatives with respect to θ on both sides of (54), we obtain

$$\partial^2_{\theta\theta} h = -\frac{R\sigma^2}{2}\frac{p\partial^2_{\theta\theta}p - (\partial_\theta p)^2}{p^2} - R(a-\omega)\frac{2\pi}{p\int p^{-1}}\frac{\partial_\theta p}{p}. \tag{55}$$

Let $v = \sqrt{p}$, then

$$\begin{aligned}
\partial_\theta h &= -R\sigma^2\frac{\partial_\theta v}{v} - R(a-\omega)\left(1 - \frac{2\pi}{v^2\int v^{-2}}\right), \\
\partial^2_{\theta\theta} h &= -R\sigma^2\frac{\partial^2_{\theta\theta}v}{v} + R\sigma^2\left(\frac{\partial_\theta v}{v}\right)^2 - 2R(a-\omega)\frac{2\pi}{v^2\int v^{-2}}\frac{\partial_\theta v}{v}.
\end{aligned} \tag{56}$$

From the Assumption 2 and Eqn. (11a),

$$(\omega - a)\partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c} + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta}h. \tag{57}$$

Substituting (56) into (57), we obtain the left hand side (LHS) of (57) as

$$-R\sigma^2(\omega-a)\frac{\partial_\theta v}{v} + R(\omega-a)^2\left(1 - \frac{2\pi}{v^2\int v^{-2}}\right),$$

and the right hand side (RHS) of (57) as

$$\frac{R\sigma^4}{2}\frac{\partial^2_{\theta\theta}v}{v} + \frac{R}{2}(\omega-a)^2\left(1 - \frac{2\pi}{v^2\int v^{-2}}\right)^2 - R\sigma^2(\omega-a)\frac{\partial_\theta v}{v} + (\eta^* - \bar{c}).$$

So Eqn. (57) becomes

$$\begin{aligned}
0 &= \frac{R\sigma^4}{2}\frac{\partial^2_{\theta\theta}v}{v} + \frac{R}{2}(\omega-a)^2\left(1 - \frac{2\pi}{v^2\int v^{-2}}\right)^2 - R(\omega-a)^2\left(1 - \frac{2\pi}{v^2\int v^{-2}}\right) + (\eta^* - \bar{c}), \\
&= \frac{R\sigma^4}{2}\frac{\partial^2_{\theta\theta}v}{v} - \frac{R}{2}(\omega-a)^2\left(1 - \frac{2\pi}{v^2\int v^{-2}}\right)\left(1 + \frac{2\pi}{v^2\int v^{-2}}\right) + (\eta^* - \bar{c})
\end{aligned} \tag{58}$$

Multiplying both sides of (58) with $\frac{2v}{R\sigma^4}$, one obtains the nonlinear equation (15). Finally, (16) is just the constraint for density function $p = v^2$, and (17) is the same as (11c) under Assump 2.

(ii) First multiplying both sides of (21) with $\frac{p}{R}$ and do a partial derivative with respect to θ, one obtains

$$\begin{aligned}
\partial_\theta\left[\frac{p}{R}(\partial_\theta h)\right] &= -\frac{\sigma^2}{2}\partial^2_{\theta\theta}p + \partial_\theta\left[(\omega-a)(1 - \frac{2\pi}{p\int_0^{2\pi}p^{-1}\,\mathrm{d}\theta})p\right], \\
&= -\frac{\sigma^2}{2}\partial^2_{\theta\theta}p + (\omega-a)\partial_\theta\left[p - \frac{2\pi}{\int_0^{2\pi}p^{-1}\,\mathrm{d}\theta}\right], \\
&= -\frac{\sigma^2}{2}\partial^2_{\theta\theta}p + (\omega-a)\partial_\theta p,
\end{aligned}$$

which gives

$$(\omega - a)\partial_\theta p = \frac{1}{R}\partial_\theta\left[p(\partial_\theta h)\right] + \frac{\sigma^2}{2}\partial^2_{\theta\theta}p.$$

Since $p(\theta, t; \omega) = v^2(\theta - at; \omega)$,

$$\partial_t p + \omega\partial_\theta p = (\omega - a)\partial_\theta p = \frac{1}{R}\partial_\theta\left[p(\partial_\theta h)\right] + \frac{\sigma^2}{2}\partial^2_{\theta\theta}p,$$

which gives (11b).

Next, substitutting $p(\theta, t; \omega) = v^2(\theta - at; \omega)$ into (21), one obtains

$$\partial_\theta h = -R\sigma^2\frac{\partial_\theta v}{v} + R(\omega - a)\left(1 - \frac{2\pi}{v^2\int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\right). \tag{59}$$

So

$$\partial_t h + \omega\partial_\theta h = (\omega - a)\partial_\theta h = -R\sigma^2(\omega - a)\frac{\partial_\theta v}{v} + R(\omega - a)^2\left(1 - \frac{2\pi}{v^2\int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\right), \tag{60}$$

$$\begin{aligned}(\partial_\theta h)^2 = {} & R^2\sigma^4\left(\frac{\partial_\theta v}{v}\right)^2 + R^2(\omega - a)^2\left(1 - \frac{2\pi}{v^2\int_0^{2\pi}\,\mathrm{d}\theta}\right)^2 \\ & - 2R^2\sigma^2(\omega - a)\frac{\partial_\theta v}{v}\left(1 - \frac{2\pi}{v^2\int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\right),\end{aligned} \tag{61}$$

$$\partial^2_{\theta\theta}h = -R\sigma^2\frac{\partial^2_{\theta\theta}v}{v} + R\sigma^2\left(\frac{\partial_\theta v}{v}\right)^2 + 2R(\omega - a)\frac{2\pi}{v^2\int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\frac{\partial_\theta v}{v}. \tag{62}$$

Multiplying both sides of (61) with $\frac{1}{2R}$, those of (62) with $-\frac{\sigma^2}{2}$ and adding them together, one obtains

$$\frac{1}{2R}(\partial_\theta h)^2 - \frac{\sigma^2}{2}\partial^2_{\theta\theta}h = \frac{R\sigma^4}{2}\frac{\partial^2_{\theta\theta}v}{v} + \frac{R}{2}(\omega - a)^2\left(1 - \frac{2\pi}{v^2\int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\right)^2 - R\sigma^2(\omega - a)\frac{\partial_\theta v}{v}. \tag{63}$$

Multiplying both sides of (15) with $\frac{R\sigma^4}{2v}$, one obtains

$$\frac{R\sigma^4}{2}\frac{\partial^2_{\theta\theta}v}{v} + \eta^* - \bar{c} - \frac{R}{2}(\omega - a)^2\left(1 - \left(\frac{2\pi}{v^2\int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\right)^2\right) = 0,$$

which gives

$$\frac{R\sigma^4}{2}\frac{\partial^2_{\theta\theta}v}{v} = -(\eta^* - \bar{c}) + \frac{R}{2}(\omega - a)^2\left(1 - \left(\frac{2\pi}{v^2\int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\right)^2\right). \tag{64}$$

Substituting (64) into (63), one obtains

$$\frac{1}{2R}(\partial_\theta h)^2 - \frac{\sigma^2}{2}\partial^2_{\theta\theta}h = -(\eta^* - \bar{c}) + R(\omega - a)^2\left(1 - \frac{2\pi}{v^2\int_0^{2\pi} v^{-2}\,\mathrm{d}\theta}\right) - R\sigma^2(\omega - a)\frac{\partial_\theta v}{v},$$
$$= -(\eta^* - \bar{c}) + \partial_t h + \omega\partial_\theta h,$$

where the last equality comes from (60). Rearranging the last equation, one obtains

$$\partial_t h + \omega\partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c} + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta}h, \tag{65}$$

which gives (11a). Finally, (11c) is obtained from (17) under Assumption 2.

□

7.5 Proof of Lemma 4

Proof. The Euler-Lagrange equation (24) is obtained from considering the first variation of (22)-(23), which can be derived in a fashion similar to that in Lemma 2. Comparing equation (22) with (13), the only difference in the integrand is the first term: the latter is $\bar{c}v^2$ and the former is $\mathcal{C}[v]v^2$. So we derive its first variation here as

$$DI_1[v]\cdot v' = \int_\Omega\int_0^{2\pi} 2\mathcal{C}[v]v\cdot v'\,\mathrm{d}\theta g(\omega)\,\mathrm{d}\omega \tag{66}$$
$$+ \int_\Omega\int_0^{2\pi}\left(v^2(\theta;\omega)\int_\Omega\int_0^{2\pi} c^\bullet(\theta-\vartheta)2v(\vartheta;\omega')\cdot v'(\vartheta;\omega')\,\mathrm{d}\vartheta g(\omega')\,\mathrm{d}\omega'\right)\mathrm{d}\theta g(\omega)\,\mathrm{d}\omega \tag{67}$$
$$=: DI_{11}[v]\cdot v' + DI_{12}[v]\cdot v'. \tag{68}$$

Note the integrand of (66) is same as that of (50). Since $c^\bullet(\cdot)$ is even, (67) can be written as

$$DI_{12}[v]\cdot v' = \int_\Omega\int_0^{2\pi}\left(v^2(\theta;\omega)\int_\Omega\int_0^{2\pi} c^\bullet(\vartheta-\theta)2v(\vartheta;\omega')\cdot v'(\vartheta;\omega')\,\mathrm{d}\vartheta g(\omega')\,\mathrm{d}\omega'\right)\mathrm{d}\theta g(\omega)\,\mathrm{d}\omega \tag{69}$$
$$= \int_\Omega\int_0^{2\pi}\left(v^2(\vartheta;\omega')\int_\Omega\int_0^{2\pi} c^\bullet(\theta-\vartheta)2v(\theta;\omega)\cdot v'(\theta;\omega)\,\mathrm{d}\theta g(\omega)\,\mathrm{d}\omega\right)\mathrm{d}\vartheta g(\omega')\,\mathrm{d}\omega' \tag{70}$$
$$= \int_\Omega\int_0^{2\pi}\left(\int_\Omega\int_0^{2\pi} c^\bullet(\theta-\vartheta)v^2(\vartheta;\omega')\,\mathrm{d}\vartheta g(\omega')\,\mathrm{d}\omega'\right)2v(\theta;\omega)\cdot v'(\theta;\omega)\,\mathrm{d}\theta g(\omega)\,\mathrm{d}\omega \tag{71}$$
$$= \int_\Omega\int_0^{2\pi}\mathcal{C}[v](\theta)2v(\theta;\omega)\cdot v'(\theta;\omega)\,\mathrm{d}\theta g(\omega)\,\mathrm{d}\omega = DI_{11}[v]\cdot v', \tag{72}$$

where (70) is obtained by switching variable θ with ϑ and ω with ω' in (69), (71) is obtained by rearrangement of (70), and (72) is obtained from definition of $\mathcal{C}[v]$ in (17). So we obtain

$$DI_1[v]\cdot v' = \int_\Omega\int_0^{2\pi} 4\mathcal{C}[v]v\cdot v'\,\mathrm{d}\theta g(\omega)\,\mathrm{d}\omega,$$

where the integrand is as twice as that in (50), which leads to the difference between (24) and (15).

Multiplying both sides of (24) by $\frac{\sigma^4 R v}{2}$ and integrating from 0 to 2π, we obtain the following:

$$\begin{aligned}\lambda^*(\omega,a) &= \int_0^{2\pi} \frac{R\sigma^4}{2}(\partial_\theta v^*)^2 + 2\mathcal{C}[v^*](v^*)^2 + \frac{R}{2}(\omega-a)^2\left(1-\left(\frac{2\pi}{(v^*)^2\int(v^*)^{-2}}\right)^2\right)(v^*)^2\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{R\sigma^4}{2}(\partial_\theta v^*)^2 + 2\mathcal{C}[v^*](v^*)^2 + \frac{R}{2}(\omega-a)^2\left(1-\frac{2\pi}{(v^*)^2\int(v^*)^{-2}}\right)^2(v^*)^2\,\mathrm{d}\theta.\end{aligned}$$

Taking expectations on both sides, we obtain the result (26). □

7.6 Proof of Lemma 5

The equation (32) is re-written as

$$\sigma^4 R\partial^2_{\theta\theta} v + 2\left(\lambda - \alpha\int_0^{2\pi} c^\bullet(\theta,\vartheta)v^2(\vartheta)\,\mathrm{d}\vartheta\right)v = 0. \tag{73}$$

We substitute the expansion (36) into (73) and the normalization condition $\int v^2\,\mathrm{d}\theta = 1$, and collect the terms according to different orders of x.

At $O(1)$, we have the steady state solution

$$v_0 = \frac{1}{\sqrt{2\pi}}, \quad \lambda_0 = \alpha C_0^\bullet = \frac{\alpha}{2\pi}\int_0^{2\pi} c^\bullet(\theta-\vartheta)\,\mathrm{d}\vartheta.$$

At $O(x)$,

$$0 = \sigma^4 r_0\partial^2_{\theta\theta} v_1 + 2v_0\left(\lambda_1 - \alpha\int_0^{2\pi} c^\bullet(\theta-\vartheta)2v_0v_1(\vartheta)\,\mathrm{d}\vartheta\right), \tag{74}$$

$$0 = \int_0^{2\pi} v_1(\theta)\,\mathrm{d}\theta. \tag{75}$$

Suppose we have the Fourier expansion for the function $v_1(\theta)$

$$v_1(\theta) = \sum_k v_{1k}e^{ik\theta}. \tag{76}$$

Substitute (76) into (75),

$$\int_0^{2\pi} v_{10}\,\mathrm{d}\theta = 0 \quad \Rightarrow v_{10} = 0.$$

Substitute (76) into (74) to obtain

$$\sum_k \left(-k^2\sigma^4 r_0 - 8\alpha\pi v_0^2 C_k^\bullet\right)v_{1k}e^{ik\theta} + 2v_0\lambda_1 = 0.$$

We collect the terms with respect to $e^{ik\theta}$. When $k=0$,

$$-8\alpha\pi v_0^2 C_0^\bullet v_{10} + 2v_0\lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = 0 \text{ since } v_{10} = 0.$$

When $k=1$, $(-\sigma^4 r_0 - 8\alpha\pi v_0^2 C_1^\bullet)v_{11} = 0$. If $v_{11} \neq 0$,

$$r_0 = -\frac{8\alpha\pi v_0^2 C_1^\bullet}{\sigma^4} = \frac{\alpha}{2\sigma^4} = R_c^\alpha\,.$$

When $k \geq 2$, $C_k^\bullet = 0$,

$$-k^2\sigma^4 r_0 v_{1k} = 0 \quad \Rightarrow \quad v_{1k} = 0.$$

When $k<0$, it is similar as $k>0$. The existence of bifurcation implies $v_1 \neq 0$, so $v_{11} = \bar{v}_{1,-1} \neq 0$. So we obtain

$$v_1 = v_{11}e^{i\theta} + c.c = 2|v_{11}|\cos(\theta + \angle v_{11}), \tag{77}$$

where $|v_{11}|$ and $\angle v_{11}$ are the amplitude and phase angle, respectively, of v_{11}.

At $O(x^2)$,

$$\begin{aligned}
&\sigma^4 r_0 \partial^2_{\theta\theta} v_2 + \sigma^4 r_1 \partial^2_{\theta\theta} v_1 \\
&\quad + 2v_0\left(\lambda_2 - \alpha\int_0^{2\pi} c^\bullet(\theta-\vartheta)(v_1^2(\vartheta) + 2v_0 v_2(\vartheta))\,\mathrm{d}\vartheta\right) \\
&\quad - 4\alpha v_0 v_1(\theta)\int_0^{2\pi} c^\bullet(\theta-\vartheta)v_1(\vartheta)\,\mathrm{d}\vartheta = 0,
\end{aligned} \tag{78}$$

$$\int_0^{2\pi} v_1^2(\theta) + 2v_0 v_2(\theta)\,\mathrm{d}\theta = 0. \tag{79}$$

Suppose $v_2(\theta)$ also has the Fourier expansion

$$v_2(\theta) = \sum_k v_{2k}e^{ik\theta}. \tag{80}$$

Substitute (76) and (80) into (79),

$$v_{11}v_{1,-1} + v_0 v_{2,0} = 0, \text{ or } v_{2,0} = -\frac{v_{11}v_{1,-1}}{v_0}. \tag{81}$$

Substitute (76) and (80) into (78),

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}\Bigg\{&(-k^2\sigma^4 r_0 - 8\alpha\pi v_0^2 C_k^\bullet)v_{2k} - 4\alpha\pi v_0 C_k^\bullet\left(\sum_{m+l=k} v_{1m}v_{1l}\right) \\
&-k^2\sigma^4 r_1 v_{1k} - 8\alpha\pi v_0\left(\sum_{m+l=k} v_{1m}v_{1l}C_l^\bullet\right)\Bigg\}e^{ik\theta} + 2v_0\lambda_2 = 0.
\end{aligned}$$

We collect the terms of $e^{ik\theta}$ for different values of k. When $k=0$,

$$\begin{aligned}
0 &= -8\alpha\pi v_0^2 C_0^\bullet v_{20} - 4\alpha\pi v_0 C_0^\bullet (2v_{11}v_{1,-1}) \\
&\quad - 8\alpha\pi v_0 \left(v_{11}v_{1,-1}C_{-1}^\bullet + v_{1,-1}v_{11}C_1^\bullet\right) + 2v_0\lambda_2, \\
\Rightarrow \quad & - 8\alpha\pi v_0^2 C_0^\bullet v_{2,0} + 2v_0\lambda_2 = 0, \\
\Rightarrow \quad & \lambda_2 = \frac{8\alpha\pi v_0^2 C_0^\bullet v_{2,0}}{2v_0} = 4\alpha\pi v_0 C_0^\bullet \left(-\frac{v_{11}v_{1,-1}}{v_0}\right) \\
&\quad = -4\alpha\pi C_0^\bullet v_{11}v_{1,-1} = -\alpha\pi v_{11}v_{1,-1}
\end{aligned}$$

When $k=1$, $-\sigma^4 r_1 v_{11} = 0, \quad \Rightarrow \quad r_1 = 0$. When $k=2$,

$$\begin{aligned}
-4\sigma^4 r_0 v_{2,2} - 8\alpha\pi v_0 v_{11}^2 C_1^\bullet = 0, \quad \Rightarrow \quad v_{2,2} &= -\frac{2\alpha\pi v_0}{\sigma^4 r_0} v_{11}^2 C_1^\bullet \\
&= \tfrac{1}{2}\pi v_0 v_{11}^2.
\end{aligned}$$

When $k>2$, $v_{2k}=0$. For $k<0$, it is similar. So we obtain

$$\begin{aligned}
v_2 &= \tfrac{1}{2}v_{20} + v_{21}e^{i\vartheta} + v_{22}e^{i2\theta} + c.c \\
&= v_{20} + v_0\pi|v_{11}|^2\cos 2(\theta + \angle v_{11}) + 2|v_{21}|\cos(\theta + \angle v_{21})
\end{aligned} \tag{82}$$

At $O(x^3)$,

$$\begin{aligned}
&\sigma^4 r_0 \partial_{\theta\theta}^2 v_3 + \sigma^4 r_2 \partial_{\theta\theta}^2 v_1 \\
&+2v_0\left(\lambda_3 - \alpha\int c^\bullet(\theta-\vartheta)(2v_0v_3(\vartheta) + 2v_1(\vartheta)v_2(\vartheta))\,\mathrm{d}\vartheta\right) \\
&+2v_1(\theta)\left(\lambda_2 - \alpha\int c^\bullet(\theta-\vartheta)(v_1^2(\theta) + 2v_0v_2(\vartheta))\,\mathrm{d}\vartheta\right) \\
&-4\alpha v_0 v_2(\theta)\int c^\bullet(\theta-\vartheta)v_1(\vartheta)\,\mathrm{d}\vartheta = 0,
\end{aligned} \tag{83}$$

$$\int v_0v_3(\theta) + v_1(\theta)v_2(\theta)\,\mathrm{d}\theta = 0. \tag{84}$$

Suppose $v_3(\theta)$ has the Fourier expansion

$$v_3(\theta) = \sum_k v_{3k}e^{ik\theta}. \tag{85}$$

Substitute (76), (80) and (85) into (84),

$$v_0v_{3,0} + v_{11}v_{2,-1} + v_{1,-1}v_{2,1} = 0.$$

Substitute (76), (80) and (85) into (84),

$$
\begin{aligned}
\sum_k \Bigg\{ & (-k^2)\sigma^4 r_0 v_{3k} - \sigma^4 r_2 k^2 v_{1k} + 2\lambda_2 v_{1k} \\
& - 8\alpha\pi v_0 C^\bullet_k \left(\sum_{m+l=k} v_{1m} v_{2l} + v_0 v_{3k} \right) \\
& - 4\alpha\pi \sum_{m+l=k} v_{1m} C^\bullet_l \left(\sum_{a+b=l} v_{1a} v_{1b} + 2v_0 v_{2l} \right) \\
& - 8\alpha\pi v_0 \sum_{m+l=k} v_{2m} C^\bullet_l v_{1l} \Bigg\} e^{ik\theta} + 2v_0\lambda_3 = 0.
\end{aligned} \tag{86}
$$

We collect the terms of $e^{ik\theta}$ for different values of k. When $k = 0$,

$$
\begin{aligned}
0 = & - 8\alpha\pi v_0 C^\bullet_0 (v_{11} v_{2,-1} + v_{1,-1} v_{21} + v_0 v_{30}) \\
& - 4\alpha\pi \left(v_{11} C^\bullet_{-1} (2v_0 v_{2,-1}) + v_{1,-1} C^\bullet_1 2 v_0 v_{21} \right) \\
& - 8\alpha\pi v_0 (v_{21} C^\bullet_{-1} v_{1,-1} + v_{2,-1} C^\bullet_1 v_{11} + 2v_0 \lambda_3, \\
\Rightarrow \quad & \lambda_3 = \alpha\pi (v_{11} v_{2,-1} + v_{1,-1} v_{2,1}).
\end{aligned}
$$

When $k = 1$,

$$
\begin{aligned}
0 = & - \sigma^4 r_0 v_{31} - \sigma^4 r_2 v_{11} + 2\lambda_2 v_{11} \\
& - 8\alpha\pi v_0 (2C^\bullet_1 v_{11} v_{20} + 2C^\bullet_1 v_{1,-1} v_{22} + C^\bullet_1 v_0 v_{31}), \\
\Rightarrow \quad & r_2 = -\frac{7\alpha}{2\sigma^4} \pi v_{11} v_{1,-1}.
\end{aligned}
$$

In all, we obtain the formula

$$
R = r_0 + x^2 r_2 + o(x^2) = r_0 - \frac{7\alpha}{2\sigma^4} \pi |v_{11}|^2 x^2 + o(x^2), \tag{87}
$$

$$
\lambda = \lambda_0 + x^2 \lambda_2 + o(x^2) = \lambda_0 - \alpha\pi |v_{11}|^2 x^2 + o(x^2), \tag{88}
$$

$$
\begin{aligned}
v &= v_0 + x v_1 + x^2 v_2 + o(x^2) \\
&= v_0 + x |v_{11}| 2\cos(\theta + \angle v_{11}) + x^2 (-\sqrt{2\pi} |v_{11}|^2 \\
&\quad + \pi v_0 |v_{11}|^2 \cos 2(\theta + \angle v_{11}) + 2|v_{21}| \cos(\theta + \angle v_{11})) + o(x^2) \\
&= v_0 + 2\cos(\theta + \angle v_{11}) |v_{11}| x + \\
&\quad \left(-\sqrt{2\pi} + v_0 \pi \cos 2(\theta + \angle v_{11}) \right) |v_{11}|^2 x^2 + o(x^2).
\end{aligned} \tag{89}
$$

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