ECE 580, Optimization by Vector Space Methods Assignment # 1

Issued: January 16 Due: January 30, 2008

Reading Assignment:

Luenberger, Chapters 1 & 2.

Reminder:

Class is cancelled Jan. 21 & 23.

Problems:

1 Prove that the union of any number of open sets is open, and that the intersection of a finite number of open sets is open

2 Let X be a normed vector space, and let X^n denote the vector space consisting of n vectors of the form $(x_1, \dots, x_n)^T$. For given $y \in \mathsf{X}$ and $x \in \mathsf{X}^n$, prove that there is a vector $a^* \in \mathbb{R}^n$ that achieves the minimum,

$$\min_a \|y - a^T x\|$$

3 The normed vector space X is called *strictly normed* if $\|x + y\| = \|x\| + \|y\|$ implies that $y = \theta$, or $x = \alpha y$ for some scalar α.

(i) Show that $L_p[0, 1]$ is strictly normed for $1 < p < \infty$.
(ii) Show by example that $L_p[0, 1]$ is not strictly normed for $p = 1$ or $p = \infty$.
(iii) Show that a^* in the previous problem is unique when X is strictly normed.

4 Let $L_1[0, 1]$ denote the vector space of integrable functions on $[0, 1]$, and $C[0, 1]$ the space of continuous functions in the supremum norm. Consider the sequence of continuous functions defined by,

$$x_n(t) = \min(1, \max(0, 1 - 2^n(t - \tfrac{1}{2})))$$

Is this sequence Cauchy in $L_1[0, 1]$? In $C[0, 1]$? Is the sequence convergent in one of these normed vector spaces?

5 For any real sequence x prove that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

6 Read about the Contraction Mapping Theorem, and prove the following corollary:

Let S be a closed subset of a Banach space. Let T be a mapping from S to X that is *expansive*: For some constant $k > 1$,

$$\|T(x) - T(y)\| \geq k\|x - y\|, \qquad x, y \in S.$$

Then T has a unique fixed point.

Hint: First prove that the mapping T has an inverse if it is expansive.