**ECE 555** **Assignment # 1**

Issued: January 20 Due: January 27, 2011

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**Reading**: CTCN Appendix A.1 and Sections 3.1-3.4, and lecture notes on *Lyapunov functions and performance approximation* — See course website,
`https://netfiles.uiuc.edu/meyn/www/spm_files/Courses/ECE555-2011/555hw.html`

## Exercises:

1 Consider the "controlled random walk" (CRW) model of a multi-server queue,

$$Q(k+1) = Q(k) - U(k) + A(k+1), \qquad k \geq 0.$$

The process $\boldsymbol{Q}$ evolves on $\mathsf{X} = \mathbb{R}_+$. The process $\boldsymbol{A}$ is i.i.d., supported on $\mathbb{R}_+$, with finite variance $\sigma_A^2$.

Assume here that the service process $\boldsymbol{U}$ is defined by the linear policy $U(k) = \delta Q(k)$, where $\delta \in (0,1)$.

For any function $g\colon \mathsf{X} \to \mathsf{X}$, we denote by $Pg$ the function defined by the conditional expectation,

$$Pg\,(x) = \mathsf{E}[g(Q(k+1)) \mid Q(k) = x]$$

Compute $Pg$ for a general quadratic function $g(x) = ax^2 + bx + c$. Based on this and your reading, do you have a guess regarding the steady-state mean queue length? Its steady-state variance?

2 Consider the two-state Markov chain with transition matrix $P = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$.

(a) Compute $\mu P$ for general $\mu$, and construct the invariant measure $\pi$.

(b) Obtain the spectral representation of $P$,

$$P = \lambda_1 v^1 \mu^1 + \lambda_2 v^2 \mu^2$$

where $\{\lambda_i\}$ are eigenvalues, $\{\mu^i\}$ are left eigenvectors (taken to be row vectors), and $\{v^i\}$ are right eigenvectors (taken to be column vectors).

(c) Find a general expression for $P^n$ based on your answer to (b). At what rate does $P^n$ tend to $\pi$?

(d) Compute the resolvent $R_\gamma = [(1+\gamma)I - P]^{-1}$ for general $\gamma > 0$.

(e) Choose a pair of two dimensional vectors $w$ and $v$; the first a column vector, and the second a row vector. The product $wv$ is then a $2 \times 2$ matrix. Compute the inverse,

$$Z = [I - (P - wv)]^{-1}$$

and verify that the row vector $vZ$ is proportional to $\pi$.