A. *Problem 1*

We will derive an upperbound for the following quantities

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} P^t \bar{c}$$

for the cases where $\bar{c} = x$ and $\bar{c} = x^2$. The former is the steady-state mean queue length and the latter is the steady-state second-moment.

To derive an upper-bound, we use the following result:

Lemma 0.1: If

$$PV \le V - \bar{c} + \eta$$

where V is a nonnegative function that is finite for each x, then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} P^t \bar{c} \le \eta. \tag{1}$$

Proof: We iterate the first inequality:

$$P^2V \le PV - P\bar{c} + P\eta \le V - \bar{c} + \eta - P\bar{c} + P\eta = V - (c + P\bar{c}) + 2\eta$$

where we use the equality $P\eta = \eta$ since η is a constant function. Repeat this, we obtain

$$P^nV \le V - \sum_{t=0}^{n-1} P^t\bar{c} + n\eta$$

On rearanging terms and divide both sides by n, we obtain

$$\frac{1}{n}\sum_{t=0}^{n-1} P^t\bar{c} \le \eta + \frac{1}{n}V - \frac{1}{n}P^nV$$

Since V is a nonnegative function, we obtain $P^nV \ge 0$. By taking the limit as $n \to \infty$, we obtain the desired inequality. ■

We now calculate Pg.

$$\begin{aligned}
Pg(x) &= \mathsf{E}[g(Q(k+1)|Q(k) = x] \\
&= E[g(x - \delta x + A)] \\
&= E[a(x - \delta x + A)^2 + b(x - \delta x + A) + c] \\
&= a(1-\delta)^2x^2 + 2a(1-\delta)xE[A] + aE[A^2] + b(1-\delta)x + bE[A] + c \\
&= a(1-\delta)^2x^2 + \big(b(1-\delta) + 2a(1-\delta)E[A]\big)x + \big(aE[A^2] + bE[A] + c\big)
\end{aligned}$$

We need to make a good choice of (a, b, c) for each case.

Case 1: $\bar{c} = x$. Take $a = 0, b = 1/\delta, c = 0$, we obtain

$$Pg(x) \le g(x) - x + \frac{1}{\delta}\mathsf{E}[A]$$

Uppon applying the inequality (1), we obtain

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} P^t \bar{c} \le \frac{1}{\delta}\mathsf{E}[A]$$

Case 2: $\bar{c} = x^2$. Take $a = \frac{1}{2\delta-\delta^2}, b = \frac{2a(1-\delta)E[A]}{\delta}, c = 0$, we obtain

$$Pg(x) \le g(x) - x^2 + \frac{1}{2\delta-\delta^2}E[A^2] + \frac{2(1-\delta)}{\delta(2\delta-\delta^2)}E[A]^2$$

Uppon applying the inequality (1), we obtain

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} P^t \bar{c} \le \frac{1}{2\delta-\delta^2}E[A^2] + \frac{2(1-\delta)}{\delta(2\delta-\delta^2)}E[A]^2$$

Note that the second moment is an upperbound on the variance, so the above bound is also an upper-bound on the variance. Besides, we can also make the *conjecture* that the steady-state variance to be

$$\frac{1}{2\delta-\delta^2}E[A^2] + \frac{2(1-\delta)}{\delta(2\delta-\delta^2)}E[A]^2 - \left(\frac{1}{\delta}\mathsf{E}[A]\right)^2 = \frac{1}{2\delta-\delta^2}\sigma_A^2$$

B. Problem 2

(a)

$$\mu P = [\frac{9}{10}\mu_1 + \frac{1}{4}\mu_2, \frac{1}{10}\mu_1 + \frac{3}{4}\mu_2]$$

We construct the invariant measure, we first solve the equations $\pi P = \pi$, $\pi_1 + \pi_2 = 1$. Substituite the above formula for μP into these equations, we have

$$\begin{aligned} \frac{9}{10}\pi_1 + \frac{1}{4}\pi_2 &= \pi_1 \\ \frac{1}{10}\pi_1 + \frac{3}{4}\pi_2 &= \pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned}$$

From this, we obtain $\pi_1 = \frac{5}{7}$, $\pi_2 = \frac{2}{7}$.

(b) By solving $\det(I - \lambda P) = 0$, we could obtain that P have eigenvalues λ_1 and $\lambda_2 = \frac{13}{20}$. By solving $\mu P = \lambda_i \mu$ for $i = 1, 2$, we obtain that a pair of the left eigenvectors of P are given by $\mu^1 = [\frac{5}{7}, \frac{2}{7}]$ and $\mu^2 = [1, -1]$. A pair of the right eigen vectors that also satisfies $\mu^1\nu^1 = 1$ and $\mu^2\nu^2 = 1$ are given by $\nu^1 = [1, 1]^T$ and $\nu^2 = [\frac{2}{7}, -\frac{5}{7}]^T$. Note that from $\lambda \neq \lambda_2$, we obtain $\mu^1\nu^2 = 0$ and $\mu^2\nu^1 = 0$. Thus, we have the following decomposition of P:

$$P = \lambda_1\nu^1\mu^1 + \lambda_2\nu^2\mu^2$$

(c) It is easy to see that

$$P^n = (\lambda_1\nu^1\mu^1 + \lambda_2\nu^2\mu^2 P)P^{n-2} = (\lambda_1^2\nu^1\mu^1 + \lambda_2^2\nu^2\mu^2)P^{n-2}.$$

By induction it is easy to see that

$$P^n = \lambda_1^n\nu^1\mu^1 + \lambda_2^n\nu^2\mu^2.$$

Since $\lambda_2 < 1$, as $n \to \infty$, the second term vanishes. Consequently,

$$\lim_{n\to\infty} P^n = \nu^1\mu^1 = \mathbf{1}\pi$$

The convergence is geometric with rate λ_2, i.e.,

$$\|P^{n+1} - \mathbf{1}\pi\| = \lambda_2\|P^n - \mathbf{1}\pi\|$$

which holds for any matrix norm.

(d) Let us take $R_\gamma = \sum_{t=0}^{\infty}(1+\gamma)^{-t+1}P^t$ and verify it satisfies $R_\gamma = [(1+\gamma)I - P]^{-1}$.

$$\begin{aligned}
[(1+\gamma)I - P]R_\gamma &= (1+\gamma)R - PR \\
&= \sum_{t=0}^{\infty}(1+\gamma)^{-t}P^t - \sum_{t=0}^{\infty}(1+\gamma)^{-t+1}P^{t+1} \\
&= (1+\gamma)^{-0}P^0 = I
\end{aligned}$$

We can also write R_γ using the spectral decomposition of P.

$$\begin{aligned}
R_\gamma &= \sum_{t=0}^{\infty}[(1+\gamma)^{-t+1}\lambda^t\nu^1\mu^1 + (1+\gamma)^{-t+1}\lambda_2^t\nu^2\mu^2] \\
&= (1+\gamma)^{-1}[(\sum_{t=0}^{\infty}(\frac{\lambda_1}{1+\gamma})^t)\nu^1\mu^1 + (\sum_{t=0}^{\infty}(\frac{\lambda_2}{1+\gamma})^t)\nu^2\mu^2 \\
&= \frac{1}{1+\gamma-\lambda_1}\nu^1\mu^1 + \frac{1}{1+\gamma-\lambda_2}\nu^2\mu^2
\end{aligned}$$

(e) Many choices will make $[I - (P - wv)]$ invertible. We choose $w = [\frac{1}{10}, \frac{1}{4}]^T$ and $v = [1, 1]$. We obtain

$$I - (P - wv) = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Thus,

$$Z = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$vZ = [\frac{5}{4}, 2] = 7\pi.$$