ECE 555 **Assignment # 2**

Issued: January 27 Due: February 3, 2011

Reading: Finish reading selected sections of CTCN, and the lecture notes, *Representation theory & Markov Chains and Stochastic Stability.*

Exercises:

3 Consider the linear state space model on $\mathbb{R}^2$,

$$X(t+1) = AX(t) + \mathcal{E}(t+1)$$

where the eigenvalues of A are strictly less than one. It is assumed that $\boldsymbol{\mathcal{E}}$ is i.i.d., with zero-mean and finite variance. A course on linear systems theory will tell you that for any matrix $Q > 0$, there is a solution to the discrete-time *Lyapunov equation* $A^{\mathrm{T}}MA = M - Q$, where the 2×2 matrix M is positive definite.

(i) Obtain a solution to Poisson's equation with $c(x) = x^{\mathrm{T}}Qx$.

(ii) Obtain a solution to Foster's criterion using the Lyapunov function $V(x) = a\log(1 + h(x))$, where $a > 0$ is a constant, and h is your solution to (i).

4 Suppose that R is a *non-negative* $N \times N$ matrix: This means that $R_{ij} \geq 0$ for each $1 \leq i \leq N$. Suppose that s is a column vector, ν is a row vector, satisfying $\nu s > 0$, and $R \geq s \otimes \nu$. That is, $R_{ij} \geq s_i \nu_j$ for each i, j. Finally, suppose that $r > 0$ is a solution to the equation,

$$\nu G s = 1 \quad \text{where} \quad G = \sum_{k=0}^{\infty} r^{-k-1}(R - s \otimes \nu)^k \tag{1}$$

Using the ideas of the lecture on January 27, show that $\mu = \nu G$ and $h = Gs$ are left and right eigenvectors of R:

$$\mu R = r\mu, \qquad Rh = rh$$

Notes: This construction is part of the celebrated theory of Perron & Frobenius — begun approximately one hundred years ago. The scalar r is the *PerronFrobenius eigenvalue.*

A typical systems application uses $R_{ij} = e^{f_i}P_{ij}$, where P is a transition matrix, and f is envisioned as a function on the state space. The eigenvalue r is a value of the moment generating function found in risk-sensitive control and in large deviations theory for Markov chains. *Why*? Because in this case,

$$R^n_{ij} = \mathsf{E}\big[\exp(\sum_{t=0}^{n-1} f(X(t)))\mathbf{1}(X(n) = j) \mid X(0) = i\big]$$

5 Suppose that there exists a function s and a probability distribution ν on X satisfying $P = s \otimes \nu$ (that is, $P(x, y) = s(x)\nu(y)$ for each $x, y \in \mathsf{X}$.) Show that $s \equiv 1$, ν is an invariant measure, and hence $\boldsymbol{X}$ is i.i.d. with marginal distribution ν.

One more ...

6 Suppose that the minorization condition holds, $P \geq s \otimes \nu$, with $s(x) \equiv \epsilon > 0$ constant. This is a form of *Doeblin's condition*[1]. Observe that $Q(x, y) := [P(x, y) - \epsilon\nu(y)]/(1 - \epsilon)$ is a transition matrix (let's stick to a countable state space for simplicity). Doeblin's condition can be equivalently expressed $P = \epsilon \mathbf{1} \otimes \nu + (1 - \epsilon)Q$, or

$$P(x, y) = \epsilon\nu(y) + (1 - \epsilon)Q(x, y), \qquad x, y \in \mathsf{X}$$

Show that $G(x, \mathsf{X}) \leq \epsilon^{-1}$ for all x, where the potential kernel is given by,

$$G = \sum_{k=0}^{\infty} (P - s \otimes \nu)^k.$$

Is there an invariant probability measure?

[1]Wolfgang Doeblin was a *remarkable person. Just 10 years ago*, his sealed manuscript was opened which revealed that he developed the theory of stochastic differential equations while fighting the Germans in WW2! His formulation predates Ito, and is also simpler.