ECE 555 **Assignment # 3**

Issued: February 11 Due: February 22, 2011

Reading: Begin Chapter 9 of CTCN.

Exercises:

7 *Reversibility* The M/M/1 queue in discrete time is defined by $Q(t+1) = [Q(t) - S(t+1) + A(t+1)]_+$, where $\boldsymbol{A}$ is i.i.d. Bernoulli, and $S(t) = 1 - A(t)$ for $t \geq 1$. Denote $\alpha = \mathsf{E}[A(t)]$, $\mu = 1 - \alpha$, and the load is $\rho = \alpha/\mu$. Consider the case of a finite waiting room, of size N,

$$Q(t+1) = [Q(t) - S(t+1) + A(t+1)]_0^N$$

where $[x]_0^N = \max(\min(x, N), 0)$, $x \in \mathbb{R}$. Then $\boldsymbol{Q}$ is a Markov chain on the finite set $\{0, \ldots, N\}$. Let P denote the transition matrix.

Verify that the chain is *reversible*: There is a probability measure π satisfying the *detailed balance equations*,

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

Note that on summing each side of this equation over x, you obtain invariance $\pi P = \pi$.

Hint: When $N = \infty$ and $\rho < 1$, we have $\pi(x) = (1-\rho)\rho^x$.

8 *Rate of convergence in value iteration* In the previous model take $\rho = \alpha/\mu = 0.95$. Work out the following using Matlab.

(i) Compute the first and second largest eigenvectors of P for a three values of N (say, $N = 5, 10, 50$).

(ii) For each of these values of N, obtain the solution to Poisson's equation with $c(x) = x$, using the value iteration algorithm. You might experiment with different initial conditions: $V_0(x) = 0$, or $V_0(x) = \frac{1}{2}(\mu - \alpha)^{-1}x^2$ (the fluid value function).

(iii) Estimate the rate of convergence λ, where $\Lambda = \log(\lambda)$ is given by,

$$\Lambda := \lim_{n\to\infty} n^{-1}\log(\|h - h_n\|)$$

How does λ compare with λ_2, the second largest eigenvalue for P?

9 *Inverse dynamic programming* Consider the controlled Markov chain, evolving on $\mathbb{R}_+$:

$$X(t+1) = X(t) - U(t) + A(t+1)\,,$$

where $\boldsymbol{A}$ is i.i.d. on $\mathbb{R}_+$, with finite variance. The input is constrained: Given $X(t) = x$, we have $U(t) \in \mathsf{U}(x)$, where $\mathsf{U}(x) = \{x : 0 \leq u \leq x\}$. Let $h(x) = x^2$, and find a function $c(x)$ and constant η^* so that the ACOE holds,

$$\min_{u\in\mathsf{U}(x)} \{c(x) + u^2 + D_u h\,(x)\} = \eta^*, \qquad x \geq 0.$$

... Please turn over

A bit of theory regarding Prob. 8 (iii): If π is invariant for this irreducible P, then you can obtain, for any n,

$$P^n = (P - 1 \otimes \pi)^n + 1 \otimes \pi$$

But then, for any x, y, z,

$$P^n(x,z) - P^n(y,z) = (P - 1 \otimes \pi)^n(x,z) - (P - 1 \otimes \pi)^n(y,z)$$

The right hand side goes to zero like λ_2^n:

$$\log(\lambda_2) = \lim_{n \to \infty} n^{-1} \log(\|(P - 1 \otimes \pi)^n\|),$$

where $\| \cdot \|$ is any matrix norm.

Extra credit `http://en.wikipedia.org/wiki/Markov_chain` "This article's introduction section may not adequately summarize its contents... "

Write a proper paragraph that a junior undergrad in psychology or civil engineering can understand, and I extend the deadline to next Tuesday. You can work in groups of two or three. Please have a look at the rest of this site and see if you can spot any howlers at this site[1].

It will be great fun to change Wikipedia's definition by next Thursday!!

... We will hit "*Markov Decision Process*@wiki" next.

[1] Google's `define:howler` gives, belly laugh: a joke that seems extremely funny ... *or* ... a glaring blunder `wordnetweb.princeton.edu/perl/webwn`