ECE 417 Identification & Adaptive Control Fall 2001

Handout: ***Spectral Densities and Linear Systems***

Wide sense stationary processes Let $\boldsymbol{u} = \{u_k : k \geq 0\}$ be a (wide sense) stationary process, so that the following limits hold.

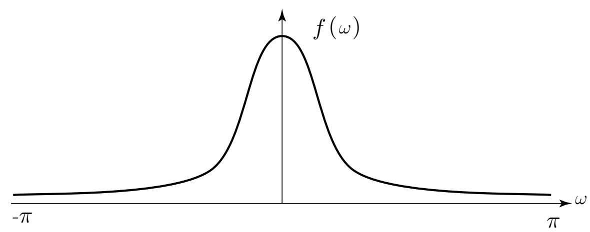
$$\mu := \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} u(k) = \mathsf{E}[u(0)]; \quad R(n) := \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}(u(k)-\mu)(u(k+n)-\mu) = \mathsf{E}[u(0)u(n)].$$

Any signal $\boldsymbol{u}$ of the form $u(k) = \sum_1^m \alpha_i \sin(\omega_i k + \phi_i)$ is WSS, but this family of signals is far larger than merely sums of sinusoids.

We call $\boldsymbol{R}$ the *autocorrelation sequence* of $\boldsymbol{u}$. The *spectral density* is defined as

$$\begin{aligned} f(\omega) &= \lim_{N\to\infty} \sum_{k=-N}^{N} R(k)e^{-jk\omega} \\ &= \text{the discrete time Fourier transform of } R(n) \end{aligned}$$

The spectral density satisfies numerous special properties:



(i) *periodic*, with period 2π;

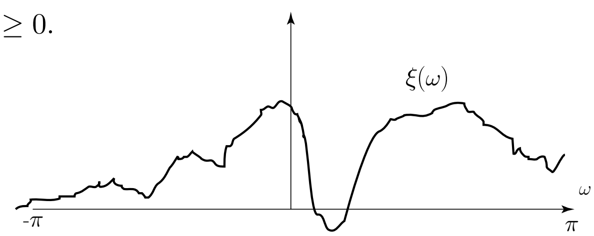
(ii) *even*, $f(\omega) = f(-\omega)$ for all ω;

(iii) *positive valued*, $f(\omega) \geq 0$ for all ω.

(iv) $R(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{jn\omega} f(\omega)\, d\omega$ (from the inversion formula for Fourier transforms)

Hence $R(n)$ is represented by its spectral density. The process $\boldsymbol{u}$ can also be represented through a *random* spectral process ξ:

(v) $u(n) = \displaystyle\int_{-\pi}^{\pi} e^{jn\omega}\, d\xi(\omega)$, $n \geq 0$.



Interpretation: The process $\boldsymbol{u}$ is approximated by a random sum of sinusoids:

$$u(n) \approx \sum_{k=-N+1}^{N} \gamma_k e^{jn\omega_k}, \qquad \text{where } \gamma_k = \xi(\omega_k) - \xi(\omega_{k-1}), \text{ and } \omega_k = k\pi/N.$$

Properties of the spectral process ξ: For $\omega_1 < \omega_2 < \omega_3$,

Orthoganl increments $\mathsf{E}[(\xi(\omega_3) - \xi(\omega_2))(\xi(\omega_2) - \xi(\omega_1))] = 0$.

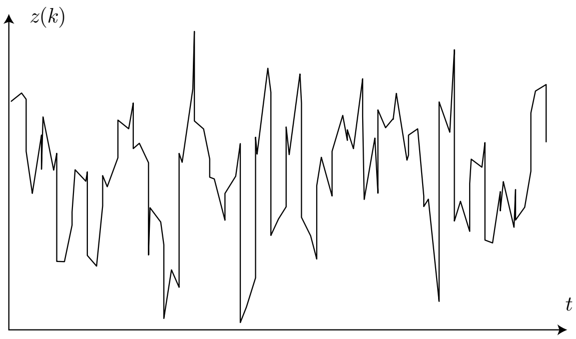
Bounded power $\mathsf{E}[|\xi(\omega_2) - \xi(\omega_1)|^2] = \dfrac{1}{2\pi}\displaystyle\int_{\omega_1}^{\omega_2} f(\tau)\, d\tau$.

Simple examples of autocorrelation and spectral density functions Note: In these examples, the mean is $\mu = 10 \neq 0$. Hence, the spectral density possesses mass at the origin, corresponding to a non-zero DC component in the signal $\boldsymbol{z}$. This will be subtracted, so that the mean of $\boldsymbol{z}$ is zero.

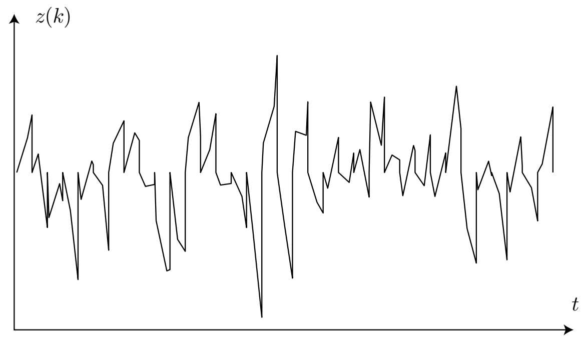
Consider the two processes,

$$z(t) = 10 + u(t) - u(t-1), \qquad z(t) = 10 + u(t) - u(t-1),$$

where $\boldsymbol{u}$ is a sequence of uncorrelated, random Normal variables with mean zero and variance one. Below are sample paths from the two models



Model 1: $z(k)$ = 10 + $u(k)$ + $u(k$-1)

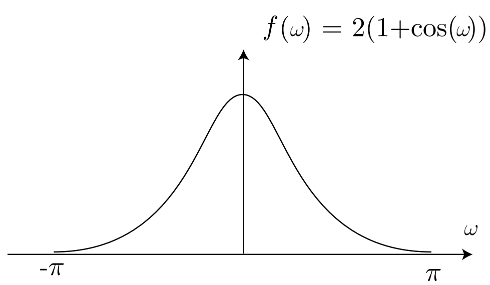


Model 2: $z(k)$ = 10 + $u(k)$ - $u(k$-1)

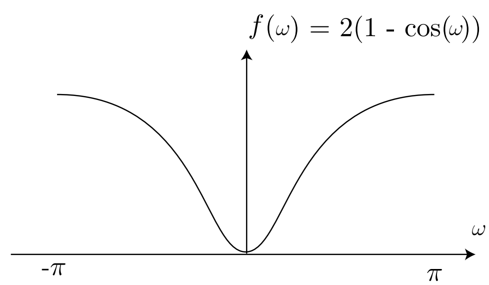
Using the definition, $R(k) = \mathsf{E}[(z(t) - \mu)(z(t+k) - \mu)]$, where $\mu = \mathsf{E}[z(t)] = 10$, we can compute the respective autocovariances:

$$R(k) = \begin{cases} 2.0 & \text{if } k = 0; \\ 1.0 & \text{if } |k| = 1; \\ 0 & \text{if } |k| \geq 2. \end{cases} \qquad R(k) = \begin{cases} 2.0 & \text{if } k = 0; \\ -1.0 & \text{if } |k| = 1; \\ 0 & \text{if } |k| \geq 2. \end{cases}$$

Using this information we may compute the spectral densities,



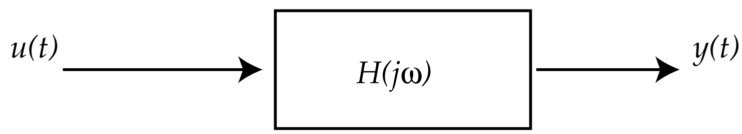
Model 1



Model 2

The first model possesses a *zero* at the frequency $\omega = \pm\pi$, while the second model possesses a zero at the DC value $\omega = 0$. The relatively smooth behavior of the sample path for the first model is reflected in the spectral densities.

Linear stochastic systems Consider the SISO system



The transfer function H is the Fourier transform of the impluse response:

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} h_n e^{-j\omega n}$$

The output y is then

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} h_n u(t-n) \\ &= \sum_{n=0}^{\infty} h_n \int_{-\pi}^{\pi} e^{j\omega(t-n)}\, d\xi_u(\omega) \\ &= \int_{-\pi}^{\pi} e^{j\omega t} H(e^{j\omega})\, d\xi_u(\omega) \end{aligned}$$

We thus have the formula $d\xi_y(\omega) = H(e^{j\omega}) d\xi_u(\omega)$. This actually means that

$$\xi_y(\omega) = \int_{-\pi}^{\omega} H(e^{j\omega})\, d\xi_u(\omega).$$

Given this formula, we can also compute the spectral density f_y:

$$f_y(\omega) = |H(e^{j\omega})|^2 f_u(\omega)$$

So, the linear system transforms signals as one would expect: frequencies are amplified or attenuated, depending on the magnitude of the frequency response. Note however that all phase information is lost in the spectral density.