Appendix A

Markov Models

This appendix describes stability theory and ergodic theory for Markov chains on a countable state space that provides foundations for the development in Part III of this book. It is distilled from Meyn and Tweedie [367], which contains an extensive bibliography (the monograph [367] is now available on-line.)

The term “chain” refers to the assumption that the time-parameter is discrete. The Markov chains that we consider evolve on a countable state space, denoted X, with transition law defined as follows,

$$P(x,y) := \mathsf{P}\{X(t+1) = y \mid X(t) = x\} \qquad x, y \in \mathsf{X},\ t = 0, 1, \dots .$$

The presentation is designed to allow generalization to more complex general state space chains as well as reflected Brownian motion models.

Since the publication of [367] there has been a great deal of progress on the theory of geometrically ergodic Markov chains, especially in the context of Large Deviations theory. See Kontoyiannis et. al. [312, 313, 311] and Meyn [364] for some recent results. The website [444] also contains on-line surveys on Markov and Brownian models.

A.1 Every process is (almost) Markov

Why do we focus so much attention on Markov chain models? An easy response is to cite the powerful analytical techniques available, such as the operator-theoretic techniques surveyed in this appendix. A more practical reply is that most processes can be approximated by a Markov chain.

Consider the following example: $\boldsymbol{Z}$ is a stationary stochastic process on the non-negative integers. A Markov chain can be constructed that has the same steady-state behavior, and similar short-term statistics. Specifically, define the probability measure on $\mathbb{Z}_+ \times \mathbb{Z}_+$ via,

$$\Pi(z_0, z_1) = \mathsf{P}\{Z(t) = z_0, Z(t+1) = z_1\}, \qquad z_0, z_1 \in \mathbb{Z}_+.$$

Note that Π captures the steady-state behavior by construction. By considering the distribution of the pair $(Z(t), Z(t+1))$ we also capture some of the dynamics of $\boldsymbol{Z}$.

The first and second marginals of Π agree, and are denoted π,

$$\pi(z_0) = \mathsf{P}\{Z(t) = z_0\} = \sum_{z_1 \in \mathbb{Z}_+} \Pi(z_0, z_1), \qquad z_0, \in \mathbb{Z}_+.$$

The transition matrix for the approximating process is defined as the ratio,

$$P(z_0, z_1) = \frac{\Pi(z_0, z_1)}{\pi(z_0)}, \qquad z_0, z_1 \in \mathsf{X},$$

with $\mathsf{X} = \{z \in \mathbb{Z}_+ : \pi(z) > 0\}$.

The following simple result is established in Chorin [111], but the origins are undoubtedly ancient. It is a component of the model reduction techniques pioneered by Mori and Zwanzig in the area of statistical mechanics [375, 505].

Proposition A.1.1. *The transition matrix P describes these aspects of the stationary process $\boldsymbol{Z}$:*

(i) One-step dynamics: $P(z_0, z_1) = \mathsf{P}\{Z(t+1) = z_1 \mid Z(t) = z_0\}$, $z_0, z_1 \in \mathsf{X}$.

(ii) Steady-state: *The probability π is invariant for P,*

$$\pi(z_1) = \sum_{z_0 \in \mathsf{X}} \pi(z_0) P(z_0, z_1), \qquad z_1, \in \mathsf{X}.$$

Proof. Part (i) is simply Baye's rule

$$\mathsf{P}\{Z(t+1) = z_1 \mid Z(t) = z_0\} = \frac{\mathsf{P}\{Z(t+1) = z_1, Z(t) = z_0\}}{\mathsf{P}\{Z(t) = z_0\}} = \frac{\Pi(z_0, z_1)}{\pi(z_0)}.$$

The definition of P gives $\pi(z_0)P(z_0, z_1) = \Pi(z_0, z_1)$, and stationarity of $\boldsymbol{Z}$ implies that $\sum_{z_0} \pi(z_0) P(z_0, z_1) = \sum_{z_0} \Pi(z_0, z_1) = \pi(z_1)$, which is (ii). □

Proposition A.1.1 is just one approach to approximation. If $\boldsymbol{Z}$ is not stationary, an alternative is to redefine Π as the limit,

$$\Pi(z_0, z_1) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathsf{P}\{Z(t) = z_0, Z(t+1) = z_1\},$$

assuming this exists for each $z_0, z_1 \in \mathbb{Z}_+$. Similar ideas are used in Section 9.2.2 to prove that an optimal policy for a controlled Markov chain can be taken stationary without loss of generality.

Another common technique is to add some history to $\boldsymbol{Z}$ via,

$$X(t) := [Z(t), Z(t-1), \dots, Z(t-n_0)],$$

where $n_0 \in [1, \infty]$ is fixed. If $n_0 = \infty$ then we are including the entire history, and in this case $\boldsymbol{X}$ is Markov: For any possible value x_1 of $X(t+1)$,

$$\mathsf{P}\{X(t+1) = x_1 \mid X(t), X(t-1), \dots\} = \mathsf{P}\{X(t+1) = x_1 \mid X(t)\}$$

A.2 Generators and value functions

The main focus of the Appendix is performance evaluation, where performance is defined in terms of a cost function $c\colon \mathsf{X} \to \mathbb{R}_+$. For a Markov model there are several performance criteria that are well-motivated and are also conveniently analyzed using tools from the general theory of Markov chains:

Discounted cost For a given discount-parameter $\gamma > 0$, recall that the discounted-cost value function is defined as the sum,

$$h_\gamma(x) := \sum_{t=0}^{\infty} (1+\gamma)^{-t-1} \mathsf{E}_x[c(X(t))], \qquad X(0) = x \in \mathsf{X}. \tag{A.1}$$

Recall from (1.18) that the expectations in (A.1) can be expressed in terms of the t-step transition matrix via,

$$\mathsf{E}[c(X(t)) \mid X(0) = x] = P^t c\,(x), \qquad x \in \mathsf{X},\ t \geq 0.$$

Consequently, denoting the *resolvent* by

$$R_\gamma = \sum_{t=0}^{\infty} (1+\gamma)^{-t-1} P^t, \tag{A.2}$$

the value function (A.1) can be expressed as the “matrix-vector product”,

$$h_\gamma(x) = R_\gamma c\,(x) := \sum_{y \in \mathsf{X}} R_\gamma(x,y) c(y), \qquad x \in \mathsf{X}.$$

Based on this representation, it is not difficult to verify the following dynamic programming equation. The discounted-cost value function solves

$$\mathcal{D} h_\gamma = -c + \gamma h_\gamma, \tag{A.3}$$

where the *generator* $\mathcal{D}$ is defined as the difference operator,

$$\mathcal{D} = P - I\,. \tag{A.4}$$

The dynamic programming equation (A.3) is a first step in the development of dynamic programming for controlled Markov chains contained in Chapter 9.

Average cost The average cost is the limit supremum of the Cesaro-averages,

$$\eta_x := \limsup_{r\to\infty} \frac{1}{r} \sum_{t=0}^{r-1} \mathsf{E}_x\big[c(X(t))\big], \qquad X(0) = x \in \mathsf{X}.$$

A probability measure is called invariant if it satisfies the invariance equation,

$$\sum_{y\in \mathsf{X}} \pi(x)\mathcal{D}(x,y) = 0, \qquad x \in \mathsf{X}, \tag{A.5}$$

Under mild stability and irreducibility assumptions we find that the average cost coincides with the spatial average $\pi(c) = \sum_{x'} \pi(x')c(x')$ for each initial condition. Under these conditions the limit supremum in the definition of the average cost becomes a limit, and it is also the limit of the normalized discounted cost for vanishing discount-rate,

$$\eta_x = \pi(c) = \lim_{r\to\infty} \frac{1}{r} \sum_{t=0}^{r-1} \mathsf{E}_x\big[c(X(t))\big] = \lim_{\gamma\downarrow 0} \gamma h_\gamma(x). \tag{A.6}$$

In a queueing network model the following x^*-*irreducibility* assumption frequently holds with $x^* \in \mathsf{X}$ taken to represent a network free of customers.

Definition A.2.1. ***Irreducibility***

The Markov chain $\boldsymbol{X}$ is called

(i) x^*-*Irreducible* if $x^* \in \mathsf{X}$ satisfies for one (and hence any) $\gamma > 0$,

$$R_\gamma(x, x^*) > 0 \qquad \text{for each } x \in \mathsf{X}.$$

(ii) The chain is simply called *irreducible* if it is x^*-irreducible for each $x^* \in \mathsf{X}$.

(iii) A x^*-irreducible chain is called *aperiodic* if there exists $n_0 < \infty$ such that $P^n(x^*, x^*) > 0$ for all $n \geq n_0$.

■

When the chain is x^*-irreducibile, we find that the most convenient sample path representations of η are expressed with respect to the *first return time* τ_{x^*} to the fixed state $x^* \in \mathsf{X}$. From Proposition A.3.1 we find that η is independent of x within the support of π, and has the form,

$$\eta = \pi(c) = \Big(\mathsf{E}_{x^*}\big[\tau_{x^*}\big]\Big)^{-1} \mathsf{E}_{x^*}\Big[\sum_{t=0}^{\tau_{x^*}-1} c(X(t))\Big]. \tag{A.7}$$

Considering the function $c(x) = \mathbf{1}\{x \neq x^*\}$ gives,

Theorem A.2.1. (Kac's Theorem) *If $\boldsymbol{X}$ is x^*-irreducible then it is positive recurrent if and only if $\mathsf{E}_{x^*}[\tau_{x^*}] < \infty$. If positive recurrence holds, then letting π denote the invariant measure for $\boldsymbol{X}$, we have*

$$\pi(x^*) = (\mathsf{E}_{x^*}[\tau_{x^*}])^{-1}. \tag{A.8}$$

Total cost and Poisson's equation For a given function $c\colon \mathsf{X} \to \mathbb{R}$ with steady state mean η, denote the centered function by $\tilde{c} = c - \eta$. Poisson's equation can be expressed,

$$\mathcal{D}h = -\tilde{c} \tag{A.9}$$

The function c is called the *forcing function*, and a solution $h\colon \mathsf{X} \to \mathbb{R}$ is known as a *relative value function*. Poisson's equation can be regarded as a dynamic programming equation; Note the similarity between (A.9) and (A.3).

Under the x^*-irreducibility assumption we have various representations of the relative value function. One formula is similar to the definition (A.6):

$$h(x) = \lim_{\gamma \downarrow 0} \bigl(h_\gamma(x) - h_\gamma(x^*) \bigr), \qquad x \in \mathsf{X}. \tag{A.10}$$

Alternatively, we have a sample path representation similar to (A.7),

$$h(x) = \mathsf{E}_x \Bigl[\sum_{t=0}^{\tau_{x^*}-1} \bigl(c(X(t)) - \eta \bigr) \Bigr], \qquad x \in \mathsf{X}. \tag{A.11}$$

This appendix contains a self-contained treatment of Lyapunov criteria for stability of Markov chains to validate formulae such as (A.11). A central result known as the *Comparison Theorem* is used to obtain bounds on η or any of the value functions described above.

These stability criteria are all couched in terms of the generator for $\boldsymbol{X}$. The most basic criterion is known as condition (V3): for a function $V\colon \mathsf{X} \to \mathbb{R}_+$, a function $f\colon \mathsf{X} \to [1, \infty)$, a constant $b < \infty$, and a finite set $S \subset \mathsf{X}$,

$$\mathcal{D}V\,(x) \le -f + b\mathbf{1}_S(x), \qquad x \in \mathsf{X}\,, \tag{\textbf{V3}}$$

or equivalently,

$$\mathsf{E}[V(X(t+1)) - V(X(t)) \mid X(t) = x] \le \begin{cases} -f(x) & x \in S^c \\ -f(x) + b & x \in S. \end{cases} \tag{A.12}$$

Under this *Lyapunov drift condition* we obtain various ergodic theorems in Section A.5. The main results are summarized as follows:

Theorem A.2.2. *Suppose that $\boldsymbol{X}$ is x^*-irreducible and aperiodic, and that there exists $V\colon \mathsf{X} \to (0, \infty)$, $f\colon \mathsf{X} \to [1, \infty)$, a finite set $S \subset \mathsf{X}$, and $b < \infty$ such that Condition (V3) holds. Suppose moreover that the cost function $c\colon \mathsf{X} \to \mathbb{R}_+$ satisfies $\|c\|_f := \sup_{x \in \mathsf{X}} c(x)/f(x) \le 1$.*

Then, there exists a unique invariant measure π satisfying $\eta = \pi(c) \le b$, and the following hold:

(i) Strong Law of Large Numbers*: For each initial condition,* $\displaystyle \frac{1}{n} \sum_{t=0}^{n-1} c(X(t)) \to \eta$ *a.s. as $n \to \infty$.*

(ii) Mean Ergodic Theorem*: For each initial condition,* $\mathsf{E}_x[c(X(t))] \to \eta$ *as* $t \to \infty$.

(iii) Discounted-cost value function h_γ*: Satisfies the uniform upper bound,*

$$h_\gamma(x) \le V(x) + b\gamma^{-1}, \qquad x \in \mathsf{X}.$$

(iv) Poisson's equation h*: Satisfies, for some* $b_1 < \infty$*,*

$$|h(x) - h(y)| \le V(x) + V(y) + b_1, \qquad x, y \in \mathsf{X}.$$

□

Proof. The Law of Large numbers is given in Theorem A.5.8, and the Mean Ergodic Theorem is established in Theorem A.5.4 based on coupling $\boldsymbol{X}$ with a stationary version of the chain.

The bound $\eta \le b$ along with the bounds on h and h_γ are given in Theorem A.4.5.

□

These results are refined elsewhere in the book in the construction and analysis of algorithms to bound or approximate performance in network models.

A.3 Equilibrium equations

In this section we consider in greater detail representations for π and h, and begin to discuss existence and uniqueness of solutions to equilibrium equations.

A.3.1 Representations

Solving either equation (A.5) or (A.9) amounts to a form of inversion, but there are two difficulties. One is that the matrices to be inverted may not be finite dimensional. The other is that these matrices are *never invertable*! For example, to solve Poisson's equation (A.9) it appears that we must invert $\mathcal{D}$. However, the function f which is identically equal to one satisfies $\mathcal{D}f \equiv 0$. This means that the null-space of $\mathcal{D}$ is non-trivial, which rules out invertibility.

On iterating the formula $Ph = h - \tilde{c}$ we obtain the sequence of identities,

$$P^2 h = h - \tilde{c} - P\tilde{c} \quad \Longrightarrow \quad P^3 h = h - \tilde{c} - P\tilde{c} - P^2\tilde{c} \quad \Longrightarrow \quad \cdots .$$

Consequently, one might expect a solution to take the form,

$$h = \sum_{i=0}^{\infty} P^i \tilde{c}. \tag{A.13}$$

When the sum converges absolutely, then this function does satisfy Poisson's equation (A.9).

A representation which is more generally valid is defined by a random sum. Define the first entrance time and first return time to a state $x^* \in \mathsf{X}$ by, respectively,

$$\sigma_{x^*} = \min(t \geq 0 : X(t) = x^*) \qquad \tau_{x^*} = \min(t \geq 1 : X(t) = x^*) \qquad \text{(A.14)}$$

Proposition A.3.1 (i) is contained in [367, Theorem 10.0.1], and (ii) is explained in Section 17.4 of [367].

Proposition A.3.1. *Let $x^* \in \mathsf{X}$ be a given state satisfying $\mathsf{E}_{x^*}[\tau_{x^*}] < \infty$. Then,*

(i) *The probability distribution defined below is invariant:*

$$\pi(x) := \Big(\mathsf{E}_{x^*}[\tau_{x^*}]\Big)^{-1} \mathsf{E}_{x^*}\Big[\sum_{t=0}^{\tau_{x^*}-1} \mathbf{1}(X(t) = x)\Big], \qquad x \in \mathsf{X}. \qquad \text{(A.15)}$$

(ii) *With π defined in (i), suppose that $c\colon \mathsf{X} \to \mathbb{R}$ is a function satisfying $\pi(|c|) < \infty$. Then, the function defined below is finite-valued on $\mathsf{X}_\pi :=$ the support of π,*

$$h(x) = \mathsf{E}_x\Big[\sum_{t=0}^{\tau_{x^*}-1} \tilde{c}(X(t))\Big] = \mathsf{E}_x\Big[\sum_{t=0}^{\sigma_{x^*}} \tilde{c}(X(t))\Big] - \tilde{c}(x^*), \qquad x \in \mathsf{X}. \qquad \text{(A.16)}$$

Moreover, h solves Poisson's equation on X_π.

□

The formulae for π and h given in Proposition A.3.1 are perhaps the most commonly known representations. In this section we develop operator-theoretic representations that are truly based on matrix inversion. These representations help to simplify the stability theory that follows, and they also extend most naturally to general state-space Markov chains, and processes in continuous time.

Operator-theoretic representations are formulated in terms of the resolvent *resolvent matrix* defined in (A.2). In the special case $\gamma = 1$ we omit the subscript and write,

$$R(x, y) = \sum_{t=0}^{\infty} 2^{-t-1} P^t(x, y), \qquad x, y \in \mathsf{X}. \qquad \text{(A.17)}$$

In this special case, the resolvent satisfies $R(x, \mathsf{X}) := \sum_y R(x, y) = 1$, and hence it can be interpreted as a transition matrix. In fact, it is precisely the transition matrix for a sampled process. Suppose that $\{t_k\}$ is an i.i.d. process with geometric distribution satisfying $\mathsf{P}\{t_k = n\} = 2^{-n-1}$ for $n \geq 0$, $k \geq 1$. Let $\{T_k : k \geq 0\}$ denote the sequence of partial sums,

$$T_0 = 0, \text{ and } T_{k+1} = T_k + t_{k+1} \text{ for } k \geq 0.$$

Then, the sampled process,

$$Y(k) = X(T_k), \qquad k \geq 0, \qquad \text{(A.18)}$$

is a Markov chain with transition matrix R.

Solutions to the invariance equations for $\boldsymbol{Y}$ and $\boldsymbol{X}$ are closely related:

Proposition A.3.2. *For any Markov chain $\boldsymbol{X}$ on X with transition matrix P,*

(i) *The resolvent equation holds,*

$$\mathcal{D}R = R\mathcal{D} = \mathcal{D}_R, \qquad \text{where } \mathcal{D}_R = R - I. \tag{A.19}$$

(ii) *A probability distribution π on X is P-invariant if and only if it is R-invariant.*

(iii) *Suppose that an invariant measure π exists, and that $g\colon \mathsf{X} \to \mathbb{R}$ is given with $\pi(|g|) < \infty$. Then, a function $h\colon \mathsf{X} \to \mathbb{R}$ solves Poisson's equation $\mathcal{D}h = -\tilde{g}$ with $\tilde{g} := g - \pi(g)$, if and only if*

$$\mathcal{D}_R h = -R\tilde{g}. \tag{A.20}$$

Proof. From the definition of R we have,

$$PR = \sum_{t=0}^{\infty} 2^{-(t+1)} P^{t+1} = \sum_{t=1}^{\infty} 2^{-t} P^t = 2R - I.$$

Hence $\mathcal{D}R = PR - R = R - I$, proving (i).

To see (ii) we pre-multiply the resolvent equation (A.19) by π,

$$\pi\mathcal{D}R = \pi\mathcal{D}_R$$

Obviously then, $\pi\mathcal{D} = 0$ if and only if $\pi\mathcal{D}_R = 0$, proving (ii). The proof of (iii) is similar. □

The operator-thoretic representations of π and h are obtained under the following *minorization condition*: Suppose that $s\colon \mathsf{X} \to \mathbb{R}_+$ is a given function, and ν is a probability on X such that

$$R(x,y) \geq s(x)\nu(y) \qquad x, y \in \mathsf{X}. \tag{A.21}$$

For example, if ν denotes the probability on X which is concentrated at a singleton $x^* \in \mathsf{X}$, and s denotes the function on X given by $s(x) := R(x, x^*)$, $x \in \mathsf{X}$, then we do have the desired lower bound,

$$R(x,y) \geq R(x,y)\mathbf{1}_{x^*}(y) = s(x)\nu(y) \qquad x, y \in \mathsf{X}.$$

The inequality (A.21) is a matrix inequality that can be written compactly as,

$$R \geq s \otimes \nu \tag{A.22}$$

where R is viewed as a matrix, and the right hand side is the outer product of the column vector s, and the row vector ν. From the resolvent equation and (A.22) we can now give a roadmap for solving the invariance equation (A.5). Suppose that we already have an invariant measure π, so that

$$\pi R = \pi.$$

Then, on subtracting $s \otimes \nu$ we obtain,

$$\pi(R - s \otimes \nu) = \pi R - \pi[s \otimes \nu] = \pi - \delta\nu,$$

where $\delta = \pi(s)$. Rearranging gives,

$$\pi[I - (R - s \otimes \nu)] = \delta\nu. \tag{A.23}$$

We can now attempt an inversion. The point is, the operator $\mathcal{D}_R := I - R$ is not invertible, but by subtracting the outer product $s \otimes \nu$ there is some hope in constructing an inverse. Define the *potential matrix* as

$$G = \sum_{n=0}^{\infty} (R - s \otimes \nu)^n\,. \tag{A.24}$$

Under certain conditions we do have $G = [I - (R - s \otimes \nu)]^{-1}$, and hence from (A.23) we obtain the representation of π,

$$\pi = \delta[\nu G]. \tag{A.25}$$

We can also attempt the 'forward direction' to construct π: Given a pair s, ν satisfying the lower bound (A.22), we *define* $\mu := \nu G$. We must then answer two questions: (i) when is μ invariant? (ii) when is $\mu(\mathsf{X}) < \infty$? If both are affirmative, then we do have an invariant measure, given by

$$\pi(x) = \frac{\mu(x)}{\mu(\mathsf{X})}, \qquad x \in \mathsf{X}.$$

We will show that μ always exists as a finite-valued measure on X, and that it is always *subinvariant*,

$$\mu(y) \geq \sum_{x \in \mathsf{X}} \mu(x) R(x, y), \qquad y \in \mathsf{X}.$$

Invariance and finiteness both require some form of *stability* for the process.

The following result shows that the formula (A.25) coincides with the representation given in (A.15) for the sampled chain $\boldsymbol{Y}$.

Proposition A.3.3. *Suppose that $\nu = \delta_{x^*}$, the point mass at some state $x^* \in \mathsf{X}$, and suppose that $s(x) := R(x, x^*)$ for $x \in \mathsf{X}$. Then we have for each bounded function $g \colon \mathsf{X} \to \mathbb{R}$,*

$$(R - s \otimes \nu)^n g\,(x) = \mathsf{E}_x[g(Y(n)) \mathbf{1}\{\tau_{x^*}^{Y} > n\}], \qquad x \in \mathsf{X},\ n \geq 1, \tag{A.26}$$

where $\tau_{x^*}^{Y}$ denotes the first return time to x^* for the chain $\boldsymbol{Y}$ defined in (A.18). Consequently,

$$Gg\,(x) := \sum_{n=0}^{\infty} (R - s \otimes \nu)^n g\,(x) = \mathsf{E}_x\Big[\sum_{t=0}^{\tau_{x^*}^{Y} - 1} g(Y(t))\Big].$$

Proof. We have $(R - s \otimes \nu)(x,y) = R(x,y) - R(x,x^*)\mathbf{1}_{y=x^*} = R(x,y)\mathbf{1}_{y\neq x^*}$. Or, in probabilistic notation,

$$(R - s \otimes \nu)(x,y) = \mathsf{P}_x\{Y(1) = y, \tau^Y_{x^*} > 1\}, \qquad x, y \in \mathsf{X}.$$

This establishes the formula (A.26) for $n = 1$. The result then extends to arbitrary $n \geq 1$ by induction. If (A.26) is true for any given n, then $(R - s \otimes \nu)^{n+1}(x,g) =$

$$\sum_{y\in\mathsf{X}} \big[(R - s \otimes \nu)(x,y)\big]\big[(R - s \otimes \nu)^n(y,g)\big]$$

$$\begin{aligned}
&= \sum_{y\in\mathsf{X}} \mathsf{P}_x\{Y(1) = y, \tau^Y_{x^*} > 1\}\mathsf{E}_y[g(Y(n))\mathbf{1}\{\tau^Y_{x^*} > n\}] \\
&= \mathsf{E}_x\Big[\mathbf{1}\{\tau^Y_{x^*} > 1\}\mathsf{E}[g(Y(n+1))\mathbf{1}\{Y(t) \neq x^*,\ t = 2, \dots, n+1\} \mid Y(1)]\Big] \\
&= \mathsf{E}_x\big[g(Y(n+1))\mathbf{1}\{\tau^Y_{x^*} > n+1\}\big]
\end{aligned}$$

where the second equation follows from the induction hypothesis, and in the third equation the Markov property was applied in the form (1.19) for $\boldsymbol{Y}$. The final equation follows from the smoothing property of the conditional expectation. □

A.3.2 Communication

The following result shows that one can assume without loss of generality that the chain is irreducible by restricting to an *absorbing* subset of X. The set $\mathsf{X}_{x^*} \subset \mathsf{X}$ defined in Proposition A.3.4 is known as a *communicating class*.

Proposition A.3.4. *For each $x^* \in \mathsf{X}$ the set defined by*

$$\mathsf{X}_{x^*} = \{y : R(x^*, y) > 0\} \tag{A.27}$$

is absorbing: $P(x, \mathsf{X}_{x^*}) = 1$ for each $x \in \mathsf{X}_{x^*}$. Consequently, if $\boldsymbol{X}$ is x^*-irreducible then the process may be restricted to X_{x^*}, and the restricted process is irreducible.

Proof. We have $DR = R - I$, which implies that $R = \frac{1}{2}(RP + I)$. Consequently, for any $x_0, x_1 \in \mathsf{X}$ we obtain the lower bound,

$$R(x^*, x_1) \geq \tfrac{1}{2}\sum_{y\in\mathsf{X}} R(x^*, y)P(y, x_1) \geq \tfrac{1}{2}R(x^*, x_0)P(x_0, x_1).$$

Consequently, if $x_0 \in \mathsf{X}_{x^*}$ and $P(x_0, x_1) > 0$ then $x_1 \in \mathsf{X}_{x^*}$. This shows that X_{x^*} is always absorbing. □

The resolvent equation in Proposition A.3.2 (i) can be generalized to any one of the resolvent matrices $\{R_\gamma\}$:

Proposition A.3.5. *Consider the family of resolvent matrices (A.2). We have the two resolvent equations,*

(i) $[\gamma I - \mathcal{D}]R_\gamma = R_\gamma[\gamma I - \mathcal{D}] = I$, $\gamma > 0$.

(ii) *For distinct* $\gamma_1, \gamma_2 \in (1, \infty)$,

$$R_{\gamma_2} = R_{\gamma_1} + (\gamma_1 - \gamma_2)R_{\gamma_1}R_{\gamma_2} = R_{\gamma_1} + (\gamma_1 - \gamma_2)R_{\gamma_2}R_{\gamma_1} \tag{A.28}$$

Proof. For any $\gamma > 0$ we can express the resolvent as a matrix inverse,

$$R_\gamma = \sum_{t=0}^{\infty}(1+\gamma)^{-t-1}P^t = [\gamma I - \mathcal{D}]^{-1}, \qquad x \in \mathsf{X}, \tag{A.29}$$

and from (A.29) we deduce (i). To see (ii) write,

$$[\gamma_1 I - \mathcal{D}] - [\gamma_2 I - \mathcal{D}] = (\gamma_1 - \gamma_2)I$$

Multiplying on the left by $[\gamma_1 I - \mathcal{D}]^{-1}$ and on the right by $[\gamma_2 I - \mathcal{D}]^{-1}$ gives,

$$[\gamma_2 I - \mathcal{D}]^{-1} - [\gamma_1 I - \mathcal{D}]^{-1} = (\gamma_1 - \gamma_2)[\gamma_1 I - \mathcal{D}]^{-1}[\gamma_2 I - \mathcal{D}]^{-1}$$

which is the first equality in (A.28). The proof of the second equality is identical. □

When the chain is x^*-irreducible then one can solve the minorization condition with s positive everywhere:

Lemma A.3.6. *Suppose that* $\boldsymbol{X}$ *is* x^**-irreducible. Then there exists* $s\colon \mathsf{X} \to [0,1]$ *and a probability distribution* ν *on* X *satisfying,*

$$s(x) > 0 \textit{ for all } x \in \mathsf{X} \textit{ and } \nu(y) > 0 \textit{ for all } y \in \mathsf{X}_{x^*}.$$

Proof. Choose $\gamma_1 = 1$, $\gamma_2 \in (0,1)$, and define $s_0(x) = \mathbf{1}_{x^*}(x)$, $\quad \nu_0(y) = R_{\gamma_2}(x^*, y)$, $x, y \in \mathsf{X}$, so that $R_{\gamma_2} \geq s_0 \otimes \nu_0$. From (A.28),

$$R_{\gamma_2} = R_1 + (1 - \gamma_2)R_1 R_{\gamma_2} \geq (1 - \gamma_2)R_1[s_0 \otimes \nu_0].$$

Setting $s = (1-\gamma_2)R_1 s_0$ and $\nu = \nu_0$ gives $R = R_1 \geq s \otimes \nu$. The function s is positive everywhere due to the x^*-irreducibility assumption, and ν is positive on X_{x^*} since $R_{\gamma_2}(x^*, y) > 0$ if and only if $R(x^*, y) > 0$. □

The following is the key step in establishing subinvariance, and criteria for invariance. Note that Lemma A.3.7 (i) only requires the minorization condition (A.22).

Lemma A.3.7. *Suppose that the function* $s\colon \mathsf{X} \to [0,1)$ *and the probability distribution* ν *on* X *satisfy (A.22). Then,*

(i) $Gs\,(x) \leq 1$ *for every* $x \in \mathsf{X}$.

(ii) $(R - s \otimes \nu)G = G(R - s \otimes \nu) = G - I$.

(iii) *If* $\boldsymbol{X}$ *is* x^**-irreducible and* $s(x^*) > 0$*, then* $\sup_{x \in \mathsf{X}} G(x, y) < \infty$ *for each* $y \in \mathsf{X}$.

Proof. For $N \geq 0$, define $g_N : \mathsf{X} \to \mathbb{R}_+$ by

$$g_N = \sum_{n=0}^{N} (R - s \otimes \nu)^n s.$$

We show by induction that $g_N(x) \leq 1$ for every $x \in \mathsf{X}$ and $N \geq 0$. This will establish (i) since $g_N \uparrow Gs$, as $N \uparrow \infty$.

For each x we have $g_0(x) = s(x) = s(x)\nu(\mathsf{X}) \leq R(x, \mathsf{X}) = 1$, which verifies the induction hypothesis when $N = 0$. If the induction hypothesis is true for a given $N \geq 0$, then

$$\begin{aligned} g_{N+1}(x) &= (R - s \otimes \nu) g_N(x) + s(x) \\ &\leq (R - s \otimes \nu)\mathbf{1}(x) + s(x) \\ &= [R(x, \mathsf{X}) - s(x)\nu(\mathsf{X})] + s(x) = 1, \end{aligned}$$

where in the last equation we have used the assumption that $\nu(\mathsf{X}) = 1$.

Part (ii) then follows from the definition of G.

To prove (iii) we first apply (ii), giving $GR = G - I + Gs \otimes \nu$. Consequently, from (i),

$$GRs = Gs - s + \nu(s)Gs \leq 2 \qquad \text{on } \mathsf{X}. \tag{A.30}$$

Under the conditions of the lemma we have $Rs\,(y) > 0$ for every $y \in \mathsf{X}$, and this completes the proof of (iii), with the explicit bound,

$$G(x, y) \leq 2(Rs\,(y))^{-1} \text{ for all } x, y \in \mathsf{X}.$$

□

It is now easy to establish subinvarance:

Proposition A.3.8. *For a x^*-irreducible Markov chain, and any small pair (s, ν), the measure $\mu = \nu G$ is always subinvariant. Writing $p_{(s,\nu)} = \nu G s$, we have*

(i) $p_{(s,\nu)} \leq 1$;

(ii) *μ is invariant if and only if $p_{(s,\nu)} = 1$.*

(iii) *μ is finite if and only if $\nu G(\mathsf{X}) < \infty$.*

Proof. Result (i) follows from Lemma A.3.7 and the assumption that ν is a probability distribution on X. The final result (iii) is just a restatement of the definition of μ. For (ii), write

$$\begin{aligned} \mu R &= \sum_{n=0}^{\infty} \nu(R - s \otimes \nu)^n R \\ &= \sum_{n=0}^{\infty} \nu(R - s \otimes \nu)^{n+1} + \sum_{n=0}^{\infty} \nu(R - s \otimes \nu)^n s \otimes \nu \\ &= \mu - \nu + p_{(s,\nu)}\nu \leq \mu. \end{aligned}$$

□

It turns out that the case $p_{(s,\nu)} = 1$ is equivalent to a form of recurrence.

Definition A.3.1. ***Recurrence***

A x^*-irreducible Markov chain $\boldsymbol{X}$ is called,

(i) *Harris recurrent*, if the return time (A.14) is finite almost-surely from each initial condition,

$$\mathsf{P}_x\{\tau_{x^*} < \infty\} = 1, \qquad x \in \mathsf{X}.$$

(ii) *Positive Harris recurrent*, if it is Harris recurrent, and an invariant measure π exists.

■

For a proof of the following result the reader is referred to [388]. A key step in the proof is the application of Proposition A.3.3.

Proposition A.3.9. *Under the conditions of Proposition A.3.8,*

(i) $p_{(s,\nu)} = 1$ *if and only if* $\mathsf{P}_{x^*}\{\tau_{x^*} < \infty\} = 1$. *If either of these conditions hold then* $Gs(x) = \mathsf{P}_x\{\tau_{x^*} < \infty\} = 1$ *for each* $x \in \mathsf{X}_{x^*}$.

(ii) $\mu(\mathsf{X}) < \infty$ *if and only if* $\mathsf{E}_{x^*}[\tau_{x^*}] < \infty$. □

To solve Poisson's equation (A.9) we again apply Proposition A.3.2. First note that the solution h is not unique since we can always add a constant to obtain a new solution to (A.9). This gives us some flexibility: *assume* that $\nu(h) = 0$, so that $(R - s \otimes \nu)h = Rh$. This combined with the formula $Rh = h - Rf + \eta$ given in (A.20) leads to a familiar looking identity,

$$[I - (R - s \otimes \nu)]h = R\tilde{c}.$$

Provided the inversion can be justified, this leads to the representation

$$h = [I - (R - s \otimes \nu)]^{-1} R\tilde{c} = GR\tilde{c}. \tag{A.31}$$

Based on this we define the *fundamental matrix*,

$$Z := GR(I - \mathbf{1} \otimes \pi), \tag{A.32}$$

so that the function in (A.31) can be expressed $h = Zc$.

Proposition A.3.10. *Suppose that* $\mu(\mathsf{X}) < \infty$. *If* $c \colon \mathsf{X} \to \mathbb{R}$ *is any function satisfying* $\mu(|c|) < \infty$ *then the function* $h = Zc$ *is finite valued on the support of* ν *and solves Poisson's equation.*

Proof. We have $\mu(|\tilde{c}|) = \nu(GR|\tilde{c}|)$, which shows that $\nu(GR|\tilde{c}|) < \infty$. It follows that h is finite valued a.e. $[\nu]$. Note also from the representation of μ,

$$\nu(h) = \nu(GR\tilde{c}) = \mu(R\tilde{c}) = \mu(\tilde{c}) = 0.$$

To see that h solves Poisson's equation we write,

$$Rh = (R - s \otimes \nu)h = (R - s \otimes \nu)GR\tilde{c} = GR\tilde{c} - R\tilde{c},$$

where the last equation follows from Lemma A.3.7 (ii). We conclude that h solves the version of Poisson's equation (A.20) for the resolvent with forcing function Rc, and Proposition A.3.2 then implies that h is a solution for P with forcing function c. □

A.3.3 Near-monotone functions

A function $c\colon \mathsf{X} \to \mathbb{R}$ is called *near-monotone* if the sublevel set, $S_c(r) := \{x : c(x) \leq r\}$ is finite for each $r < \sup_{x\in\mathsf{X}} c(x)$. In applications the function c is typically a cost function, and hence the near monotone assumption is the natural condition that large states have relatively high cost.

The function $c = \mathbf{1}_{\{x^*\}^c}$ is near monotone since $S_c(r)$ consists of the singleton $\{x^*\}$ for $r \in [0,1)$, and it is empty for $r < 0$. A solution to Poisson's equation with this forcing function can be constructed based on the sample path formula (A.16),

$$\begin{aligned} h(x) &= \mathsf{E}_x\Big[\sum_{t=0}^{\tau_{x^*}-1} \mathbf{1}_{\{x^*\}^c}(X(t)) - \pi(\{x^*\}^c)\Big] \\ &= (1-\pi(\{x^*\}^c)\mathsf{E}_x[\tau_{x^*}] - \mathbf{1}_{x^*}(x) = \pi(x^*)\mathsf{E}_x[\sigma_{x^*}] \end{aligned} \tag{A.33}$$

The last equality follows from the formula $\pi(x^*)\mathsf{E}_{x^*}[\tau_{x^*}] = 1$ (see (A.15)) and the definition $\sigma_{x^*} = 0$ when $X(0) = x^*$.

The fact that h is bounded from below is a special case of the following general result.

Proposition A.3.11. *Suppose that c is near monotone with $\eta = \pi(c) < \infty$. Then,*

(i) *The relative value function h given in (A.31) is uniformly bounded from below, finite-valued on X_{x^*}, and solves Poisson's equation on the possibly larger set $\mathsf{X}_h = \{x \in \mathsf{X} : h(x) < \infty\}$.*

(ii) *Suppose there exists a non-negative valued function satisfying $g(x) < \infty$ for some $x \in \mathsf{X}_{x^*}$, and the Poisson inequality,*

$$\mathcal{D}g\,(x) \leq -c(x) + \eta, \qquad x \in \mathsf{X}. \tag{A.34}$$

Then $g(x) = h(x) + \nu(g)$ for $x \in \mathsf{X}_{x^*}$, where h is given in (A.31). Consequently, g solves Poisson's equation on X_{x^*}.

Proof. Note that if $\eta = \sup_{x \in \mathsf{X}} c(x)$ then $c(x) \equiv \eta$ on X_{x^*}, so we may take $h \equiv 1$ to solve Poisson's equation.

We henceforth assume that $\eta < \sup_{x \in \mathsf{X}} c(x)$, and define $S = \{x \in \mathsf{X} : c(x) \leq \eta\}$. This set is finite since c is near-monotone. We have the obvious bound $\tilde{c}(x) \geq -\eta \mathbf{1}_S(x)$ for $x \in \mathsf{X}$, and hence

$$h(x) \geq -\eta G R \mathbf{1}_S\,(x), \qquad x \in \mathsf{X}.$$

Lemma A.3.7 and (A.30) imply that $GR\mathbf{1}_S$ is a bounded function on X. This completes the proof that h is bounded from below, and Proposition A.3.10 establishes Poisson's equation.

To prove (ii) we maintain the notation used in Proposition A.3.10. On applying Lemma A.3.6 we can assume without loss of generality that the pair (s, ν) used in the definition of G are non-zero on X_{x^*}. Note first of all that by the resolvent equation,

$$Rg - g = R\mathcal{D}g \leq -R\tilde{c}.$$

We thus have the bound,

$$(R - s \otimes \nu)g \leq g - R\tilde{c} - \nu(g)s,$$

and hence for each $n \geq 1$,

$$0 \leq (R - s \otimes \nu)^n g \leq g - \sum_{i=0}^{n-1} (R - s \otimes \nu)^i R\tilde{c} - \nu(g) \sum_{i=0}^{n-1} (R - s \otimes \nu)^i s.$$

On letting $n \uparrow \infty$ this gives,

$$g \geq GR\tilde{c} + \nu(g)Gs = h + \nu(g)h_0,$$

where $h_0 := Gs$. The function h_0 is identically one on X_{x^*} by Proposition A.3.9, which implies that $g - \nu(g) \geq h$ on X_{x^*}. Moreover, using the fact that $\nu(h) = 0$,

$$\nu(g - \nu(g) - h) = \nu(g - \nu(g)) - \nu(h) = 0.$$

Hence $g - \nu(g) - h = 0$ a.e. $[\nu]$, and this implies that $g - \nu(g) - h = 0$ on X_{x^*} as claimed. ☐

Bounds on the potential matrix G are obtained in the following section to obtain criteria for the existence of an invariant measure as well as explicit bounds on the relative value function.

A.4 Criteria for stability

To compute the invariant measure π it is necessary to compute the mean random sum (A.15), or invert a matrix, such as through an infinite sum as in (A.24). To verify the *existence* of an invariant measure is typically far easier.

In this section we describe Foster's criterion to test for the existence of an invariant measure, and several variations on this approach which are collectively called the *Foster-Lyapunov criteria* for stability. Each of these stability conditions can be interpreted as a relaxation of the Poisson *inequality* (A.34).

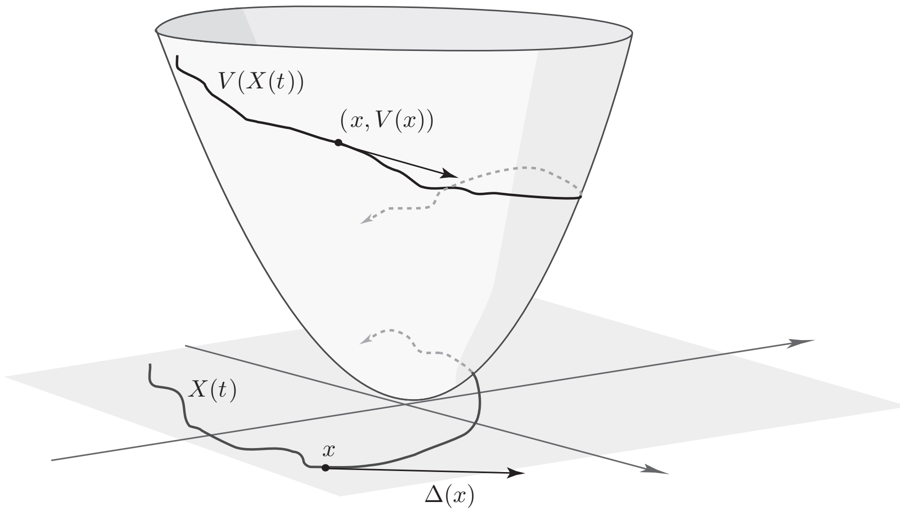

Figure A.1: $V(X(t))$ is decreasing outside of the set S.

A.4.1 Foster's criterion

Foster's criterion is the simplest of the "Foster-Lyapunov" drift conditions for stability. It requires that for a non-negative valued function V on X, a finite set $S \subset \mathsf{X}$, and $b < \infty$,

$$\mathcal{D}V(x) \leq -1 + b\mathbf{1}_S(x), \qquad x \in \mathsf{X}. \tag{\textbf{V2}}$$

This is precisely Condition (V3) (introduced at the start of this chapter) using $f \equiv 1$. The construction of the *Lyapunov function* V is illustrated using the M/M/1 queue in Section 3.3.

The existence of a solution to (V2) is equivalent to positive recurrence. This is summarized in the following.

Theorem A.4.1. (Foster's Criterion) *The following are equivalent for a x^*-irreducible Markov chain*

(i) *An invariant measure π exists.*

(ii) *There is a finite set $S \subset \mathsf{X}$ such that $\mathsf{E}_x[\tau_S] < \infty$ for $x \in S$.*

(iii) *There exists $V : \mathsf{X} \to (0, \infty]$, finite at some $x_0 \in \mathsf{X}$, a finite set $S \subset \mathsf{X}$, and $b < \infty$ such that Foster's Criterion (V2) holds.*

If (iii) holds then there exists $b_{x^*} < \infty$ such that

$$\mathsf{E}_x[\tau_{x^*}] \leq V(x) + b_{x^*}, \qquad x \in \mathsf{X}.$$

Proof. We just prove the implication (iii) $\Longrightarrow$ (i). The remaining implications may be found in [367, Chapter 11].

Take any pair (s, ν) positive on X_{x^*} and satisfying $R \geq s \otimes \nu$. On applying Proposition A.3.8 it is enough to shown that $\mu(\mathsf{X}) < \infty$ with $\mu = \nu G$.

Letting $f \equiv 1$ we have under (V2) $\mathcal{D}V \leq -f + b\mathbf{1}_S$, and on applying R to both sides of this inequality we obtain using the resolvent equation (A.19), $(R - I)V = R\mathcal{D}V \leq -Rf + bR\mathbf{1}_S$, or on rearranging terms,

$$RV \leq V - Rf + bR\mathbf{1}_S. \tag{A.35}$$

From (A.35) we have $(R - s\otimes\nu)V \leq V - Rf + g$, where $g := bR\mathbf{1}_S$. On iterating this inequality we obtain,

$$\begin{aligned}(R - s \otimes \nu)^2 V &\leq (R - s \otimes \nu)(V - Rf + g) \\ &\leq V - Rf + g \\ &\quad -(R - s \otimes \nu)Rf \\ &\quad +(R - s \otimes \nu)g.\end{aligned}$$

By induction we obtain for each $n \geq 1$,

$$0 \leq (R - s \otimes \nu)^n V \leq V - \sum_{i=0}^{n-1} (R - s \otimes \nu)^i Rf + \sum_{i=0}^{n-1} (R - s \otimes \nu)^i g\,.$$

Rearranging terms then gives,

$$\sum_{i=0}^{n-1} (R - s \otimes \nu)^i Rf \leq V + \sum_{i=0}^{n-1} (R - s \otimes \nu)^i g,$$

and thus from the definition (A.24) we obtain the bound,

$$GRf \leq V + Gg. \tag{A.36}$$

To obtain a bound on the final term in (A.36) recall that $g := bR\mathbf{1}_S$. From its definition we have,

$$GR = G[R - s \otimes \nu] + G[s \otimes \nu] = G - I + (Gs) \otimes \nu,$$

which shows that

$$Gg = bGR\mathbf{1}_S \leq b[G\mathbf{1}_S + \nu(S)Gs].$$

This is uniformly bounded over X by Lemma A.3.7. Since $f \equiv 1$ the bound (A.36) implies that $GRf\,(x) = G(x, \mathsf{X}) \leq V(x) + b_1$, $x \in \mathsf{X}$, with b_1 an upper bound on Gg.

Integrating both sides of the bound (A.36) with respect to ν gives,

$$\mu(\mathsf{X}) = \sum_{x \in \mathsf{X}} \nu(x) G(x, \mathsf{X}) \leq \nu(V) + \nu(g).$$

The minorization and the drift inequality (A.35) give

$$s\nu(V) = (s \otimes \nu)(V) \leq RV \leq V - 1 + g,$$

which establishes finiteness of $\nu(V)$, and the bound,

$$\nu(V) \leq \inf_{x\in\mathsf{X}} \frac{V(x) - 1 + g(x)}{s(x)}.$$

□

The following result illustrates the geometric considerations that may be required in the construction of a Lyapunov function, based on the relationship between the gradient $\nabla V(x)$, and the *drift vector field* $\Delta\colon \mathsf{X} \to \mathbb{R}^\ell$ defined by

$$\Delta(x) := \mathsf{E}[X(t+1) - X(t) \mid X(t) = x], \qquad x \in \mathsf{X}. \tag{A.37}$$

This geometry is illustrated in Figure A.1 based on the following proposition.

Proposition A.4.2. *Consider a Markov chain on $\mathsf{X} \subset \mathbb{Z}_+^\ell$, and a C^1 function $V\colon \mathbb{R}^\ell \to \mathbb{R}_+$ satisfying the following conditions:*

(a) *The chain is skip-free in the mean, in the sense that*

$$b_X := \sup_{x\in\mathsf{X}} \mathsf{E}[\|X(t+1) - X(t)\| \mid X(t) = x] < \infty;$$

(b) *There exists $\varepsilon_0 > 0$, $b_0 < \infty$, such that,*

$$\langle \Delta(y), \nabla V(x)\rangle \leq -(1+\varepsilon_0) + b_0(1+\|x\|)^{-1}\|x-y\|, \qquad x, y \in \mathsf{X}. \tag{A.38}$$

Then the function V solves Foster's criterion (V2).

Proof. This is an application of the Mean Value Theorem which asserts that there exists a state $\bar{X} \in \mathbb{R}^\ell$ on the line segment connecting $X(t)$ and $X(t+1)$ with,

$$V(X(t+1)) = V(X(t)) + \langle \nabla V(\bar{X}), (X(t+1) - X(t))\rangle,$$

from which the following bound follows:

$$V(X(t+1)) \leq V(X(t)) - (1+\varepsilon_0) + b_0(1+\|X(t)\|)^{-1}\|X(t+1) - X(t)\|$$

Under the skip-free assumption this shows that

$$\mathcal{D}V(x) = \mathsf{E}[V(X(t+1) - V(X(t)) \mid X(t) = x] \leq -(1+\varepsilon_0) + b_0(1+\|x\|)^{-1} b_X, \qquad \|x\| \geq n_0.$$

Hence Foster's Criterion is satisfied with the finite set, $S = \{x \in \mathsf{X} : (1+\|x\|)^{-1} b_X \geq \varepsilon_0\}$. □

A.4.2 Criteria for finite moments

We now turn to the issue of performance bounds based on the discounted-cost defined in (A.2) or the average cost $\eta = \pi(c)$ for a cost function $c\colon \mathsf{X} \to \mathbb{R}_+$. We also introduce martingale methods to obtain performance bounds. We let $\{\mathcal{F}_t : t \geq 0\}$ denote the filtration, or history generated by the chain,

$$\mathcal{F}_t := \sigma\{X(0), \dots, X(t)\}, \qquad t \geq 0.$$

Recall that a random variable τ taking values in $\mathbb{Z}_+$ is called a *stopping time* if for each $t \geq 0$,

$$\{\tau = t\} \in \mathcal{F}_t.$$

That is, by observing the process $\boldsymbol{X}$ on the time interval $[0, t]$ it is possible to determine whether or not $\tau = t$.

The Comparison Theorem is the most common approach to obtaining bounds on expectations involving stopping times.

Theorem A.4.3. (Comparison Theorem) *Suppose that the non-negative functions V, f, g satisfy the bound,*

$$\mathcal{D}V \leq -f + g. \qquad x \in \mathsf{X}. \tag{A.39}$$

Then for each $x \in \mathsf{X}$ and any stopping time τ we have

$$\mathsf{E}_x\Big[\sum_{t=0}^{\tau-1} f(X(t))\Big] \leq V(x) + \mathsf{E}_x\Big[\sum_{t=0}^{\tau-1} g(X(t))\Big].$$

Proof. Define $M(0) = V(X(0))$, and for $n \geq 1$,

$$M(n) = V(X(n)) + \sum_{t=0}^{n-1} (f(X(t)) - g(X(t))).$$

The assumed inequality can be expressed,

$$\mathsf{E}[V(X(t+1)) \mid \mathcal{F}_t] \leq V(X(t)) - f(X(t)) + g(X(t)), \qquad t \geq 0,$$

which shows that the stochastic process $\boldsymbol{M}$ is a *super-martingale*,

$$\mathsf{E}[M(n+1) \mid \mathcal{F}_n] \leq M(n), \qquad n \geq 0.$$

Define for $N \geq 1$,

$$\tau^N = \min\{t \leq \tau : t + V(X(t)) + f(X(t)) + g(X(t)) \geq N\}.$$

This is also a stopping time. The process $\boldsymbol{M}$ is uniformly bounded below by $-N^2$ on the time-interval $(0, \dots, \tau^N - 1)$, and it then follows from the super-martingale property that

$$\mathsf{E}[M(\tau^N)] \leq \mathsf{E}[M(0)] = V(x), \qquad N \geq 1.$$

From the definition of $\boldsymbol{M}$ we thus obtain the desired conclusion with τ replaced by τ^N: For each initial condition $X(0) = x$,

$$\mathsf{E}_x\Big[\sum_{t=0}^{\tau^N-1} f(X(t))\Big] \leq V(x) + \mathsf{E}_x\Big[\sum_{t=0}^{\tau^N-1} g(X(t))\Big].$$

The result then follows from the Monotone Convergence Theorem since we have $\tau^N \uparrow \tau$ as $N \to \infty$. □

In view of the Comparison Theorem, to bound $\pi(c)$ we search for a solution to (V3) or (A.39) with $|c| \leq f$. The existence of a solution to either of these drift inequalities is closely related to the following stability condition,

Definition A.4.1. ***Regularity***

Suppose that $\boldsymbol{X}$ is a x^*-irreducible Markov chain, and that $c\colon \mathsf{X} \to \mathbb{R}_+$ is a given function. The chain is called *c-regular* if the following *cost over a y-cycle* is finite for each initial condition $x \in \mathsf{X}$, and each $y \in \mathsf{X}_{x^*}$:

$$\mathsf{E}_x\Big[\sum_{t=0}^{\tau_y-1} c(X(t))\Big] < \infty.$$

■

Proposition A.4.4. *Suppose that the function $c\colon \mathsf{X} \to \mathbb{R}$ satisfies $c(x) \geq 1$ outside of some finite set. Then,*

(i) *If $\boldsymbol{X}$ is c-regular then it is positive Harris recurrent and $\pi(c) < \infty$.*

(ii) *Conversely, if $\pi(c) < \infty$ then the chain restricted to the support of π is c-regular.*

Proof. The result follows from [367, Theorem 14.0.1]. To prove (i) observe that $\boldsymbol{X}$ is Harris recurrent since $\mathsf{P}_x\{\tau_{x^*} < \infty\} = 1$ for all $x \in \mathsf{X}$ when the chain is c-regular. We have positivity and $\pi(c) < \infty$ based on the representation (A.15). □

Criteria for c-regularity will be established through operator manipulations similar to those used in the proof of Theorem A.4.1 based on the following refinement of Foster's Criterion: For a non-negative valued function V on X, a finite set $S \subset \mathsf{X}$, $b < \infty$, and a function $f\colon \mathsf{X} \to [1, \infty)$,

$$\mathcal{D}V(x) \leq -f(x) + b\mathbf{1}_S(x), \qquad x \in \mathsf{X}. \tag{\textbf{V3}}$$

The function f is interpreted as a bounding function. In Theorem A.4.5 we consider $\pi(c)$ for functions c bounded by f in the sense that,

$$\|c\|_f := \sup_{x \in \mathsf{X}} \frac{|c(x)|}{f(x)} < \infty. \tag{A.40}$$

Theorem A.4.5. *Suppose that $\boldsymbol{X}$ is x^*-irreducible, and that there exists $V : \mathsf{X} \to (0,\infty)$, $f : \mathsf{X} \to [1,\infty)$, a finite set $S \subset \mathsf{X}$, and $b < \infty$ such that (V3) holds. Then for any function $c : \mathsf{X} \to \mathbb{R}_+$ satisfying $\|c\|_f \le 1$,*

(i) *The average cost satisfies the uniform bound,*

$$\eta_x = \pi(c) \le b < \infty, \qquad x \in \mathsf{X}.$$

(ii) *The discounted-cost value function satisfies the following uniform bound, for any given discount parameter $\gamma > 0$,*

$$h_\gamma(x) \le V(x) + b\gamma^{-1}, \qquad x \in \mathsf{X}.$$

(iii) *There exists a solution to Poisson's equation satisfying, for some $b_1 < \infty$,*

$$h(x) \le V(x) + b_1, \qquad x \in \mathsf{X}.$$

Proof. Observe that (ii) and the definition (A.6) imply (i).

To prove (ii) we apply the resolvent equation,

$$PR_\gamma = R_\gamma P = (1+\gamma)R_\gamma - I. \tag{A.41}$$

Equation (A.41) is a restatement of Equation (A.29). Consequently, under (V3),

$$(1+\gamma)R_\gamma V - V = R_\gamma PV \le R_\gamma[V - f + b\mathbf{1}_S].$$

Rearranging terms gives $R_\gamma f + \gamma R_\gamma V \le V + bR_\gamma \mathbf{1}_S$. This establishes (ii) since $R_\gamma \mathbf{1}_S(x) \le R_\gamma(x,\mathsf{X}) \le \gamma^{-1}$ for $x \in \mathsf{X}$.

We now prove (iii). Recall that the measure $\mu = \nu G$ is finite and invariant since we may apply Theorem A.4.1 when the chain is x^*-irreducible. We shall prove that the function $h = GR\tilde{c}$ given in (A.31) satisfies the desired upper bound.

The proof of the implication (iii) $\Longrightarrow$ (i) in Theorem A.4.1 was based upon the bound (A.36),

$$GRf \le V + Gg,$$

where $g := bR\mathbf{1}_S$. Although it was assumed there that $f \equiv 1$, the same steps lead to this bound for general $f \ge 1$ under (V3). Consequently, since $0 \le c \le f$,

$$GR\tilde{c} \le GRf \le V + Gg.$$

Part (iii) follows from this bound and Lemma A.3.7 with $b_1 := \sup Gg\,(x) < \infty$. □

Proposition A.4.2 can be extended to provide the following criterion for finite moments in a skip-free Markov chain:

Proposition A.4.6. *Consider a Markov chain on $\mathsf{X} \subset \mathbb{R}^\ell$, and a C^1 function $V : \mathbb{R}^\ell \to \mathbb{R}_+$ satisfying the following conditions:*

(i) *The chain is skip-free in mean-square:*

$$b_{X2} := \sup_{x\in \mathsf{X}} \mathsf{E}[\|X(t+1) - X(t)\|^2 \mid X(t) = x] < \infty;$$

(ii) *There exists* $b_0 < \infty$ *such that,*

$$\langle \Delta(y), \nabla V(x)\rangle \le -\|x\| + b_0\|x-y\|^2, \qquad x,y\in\mathsf{X}. \tag{A.42}$$

Then the function V *solves (V3) with* $f(x) = 1 + \frac{1}{2}\|x\|$. □

A.4.3 State-dependent drift

In this section we consider consequences of state-dependent drift conditions of the form

$$\sum_{y\in\mathsf{X}} P^{n(x)}(x,y)V(y) \le g[V(x), n(x)], \qquad x\in S^c, \tag{A.43}$$

where $n(x)$ is a function from X to $\mathbb{Z}_+$, g is a function depending on which type of stability we seek to establish, and S is a finite set.

The function $n(x)$ here provides the state-dependence of the drift conditions, since from any x we must wait $n(x)$ steps for the drift to be negative.

In order to develop results in this framework we work with a sampled chain $\widehat{\boldsymbol{X}}$. Using $n(x)$ we define the new transition law $\{\widehat{P}(x,A)\}$ by

$$\widehat{P}(x,A) = P^{n(x)}(x,A), \qquad x\in\mathsf{X},\ A\subset\mathsf{X}, \tag{A.44}$$

and let $\widehat{\boldsymbol{X}}$ denote a Markov chain with this transition law. This Markov chain can be constructed explicitly as follows. The time $n(x)$ is a (trivial) stopping time. Let $\{n_k\}$ denote its iterates: That is, along any sample path, $n_0 = 0$, $n_1 = n(x)$ and

$$n_{k+1} = n_k + n(X(n_k)).$$

Then it follows from the strong Markov property that

$$\widehat{X}(k) = X(n_k), \qquad k\ge 0 \tag{A.45}$$

is a Markov chain with transition law $\widehat{P}$.

Let $\widehat{\mathcal{F}}_k = \mathcal{F}_{n_k}$ be the σ-field generated by the events "before n_k": that is,

$$\widehat{\mathcal{F}}_k := \{A : A\cap\{n_k\le n\}\in\mathcal{F}_n, n\ge 0\}.$$

We let $\hat\tau_S$ denote the first return time to S for the chain $\widehat{\boldsymbol{X}}$. The time n_k and the event $\{\hat\tau_S\ge k\}$ are $\widehat{\mathcal{F}}_{k-1}$-measurable for any $S\subset\mathsf{X}$.

The integer $n_{\hat\tau_S}$ is a particular time at which the original chain visits the set S. Minimality implies the bound,

$$n_{\hat\tau_S} \ge \tau_S. \tag{A.46}$$

By adding the lengths of the sampling times n_k along a sample path for the sampled chain, the time $n_{\hat{\tau}_S}$ can be expressed as the sum,

$$n_{\hat{\tau}_S} = \sum_{k=0}^{\hat{\tau}_S - 1} n(\widehat{X}(k)). \tag{A.47}$$

These relations enable us to first apply the drift condition (A.43) to bound the index at which $\widehat{\boldsymbol{X}}$ reaches S, and thereby bound the hitting time for the original chain.

We prove here a state-dependent criterion for positive recurrence. Generalizations are described in the Notes section in Chapter 10, and Theorem 10.0.1 contains strengthened conclusions for the CRW network model.

Theorem A.4.7. *Suppose that $\boldsymbol{X}$ is a x^*-irreducible chain on X, and let $n(x)$ be a function from X to $\mathbb{Z}_+$. The chain is positive Harris recurrent if there exists some finite set S, a function $V\colon \mathsf{X} \to \mathbb{R}_+$, and a finite constant b satisfying*

$$\sum_{y \in \mathsf{X}} P^{n(x)}(x,y)V(y) \le V(x) - n(x) + b\mathbf{1}_S(x), \qquad x \in \mathsf{X} \tag{A.48}$$

in which case for all x

$$\mathsf{E}_x[\tau_S] \le V(x) + b. \tag{A.49}$$

Proof. The state-dependent drift criterion for positive recurrence is a direct consequence of the f-regularity results of Theorem A.4.3, which tell us that without any irreducibility or other conditions on $\boldsymbol{X}$, if f is a non-negative function and

$$\sum_{y \in \mathsf{X}} P(x,y)V(y) \le V(x) - f(x) + b\mathbf{1}_S(x), \qquad x \in \mathsf{X} \tag{A.50}$$

for some set S then for each $x \in \mathsf{X}$

$$\mathsf{E}_x\Big[\sum_{t=0}^{\tau_S - 1} f(X(t))\Big] \le V(x) + b. \tag{A.51}$$

We now apply this result to the chain $\widehat{\boldsymbol{X}}$ defined in (A.45). From (A.48) we can use (A.51) for $\widehat{\boldsymbol{X}}$, with $f(x)$ taken as $n(x)$, to deduce that

$$\mathsf{E}_x\Big[\sum_{k=0}^{\hat{\tau}_S - 1} n(\widehat{X}(k))\Big] \le V(x) + b. \tag{A.52}$$

Thus from (A.46,A.47) we obtain the bound (A.49). Theorem A.4.1 implies that $\boldsymbol{X}$ is positive Harris. □

A.5 Ergodic theorems and coupling

The existence of a Lyapunov function satisfying (V3) leads to the ergodic theorems (1.23), and refinements of this drift inequality lead to stronger results. These results are based on the coupling method described next.

A.5.1 Coupling

Coupling is a way of comparing the behavior of the process of interest $\boldsymbol{X}$ with another process $\boldsymbol{Y}$ which is already understood. For example, if $\boldsymbol{Y}$ is taken as the stationary version of the process, with $Y(0) \sim \pi$, we then have the trivial mean ergodic theorem,

$$\lim_{t\to\infty} \mathsf{E}[c(Y(t))] = \mathsf{E}[c(Y(t_0))], \qquad t_0 \geq 0\,.$$

This leads to a corresponding ergodic theorem for $\boldsymbol{X}$ provided the two processes couple in a suitably strong sense.

To precisely define coupling we define a bivariate process,

$$\Psi(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, \qquad t \geq 0,$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are two copies of the chain with transition probability P, and different initial conditions. It is assumed throughout that $\boldsymbol{X}$ is x^*-irreducible, and we define the *coupling time* for $\boldsymbol{\Psi}$ as the first time both chains reach x^* simultaneously,

$$T = \min(t \geq 1 : X(t) = Y(t) = x^*) = \min\bigl(t : \Psi(t) = \binom{x^*}{x^*}\bigr).$$

To give a full statistical description of $\boldsymbol{\Psi}$ we need to explain how $\boldsymbol{X}$ and $\boldsymbol{Y}$ are related. We assume a form of conditional independence for $k \leq T$:

$$\begin{aligned} &\mathsf{P}\{\Psi(t+1) = (x_1, y_1)^{\mathsf{T}} \mid \Psi(0), \dots, \Psi(t); \Psi(t) = (x_0, y_0)^{\mathsf{T}},\ T > t\} \\ &\qquad\qquad\qquad\qquad = P(x_0, x_1)P(y_0, y_1). \end{aligned} \tag{A.53}$$

It is assumed that the chains coellesce at time T, so that $X(t) = Y(t)$ for $t \geq T$.

The process $\boldsymbol{\Psi}$ is not itself Markov since given $\Psi(t) = (x, x)^{\mathsf{T}}$ with $x \neq x^*$ it is impossible to know if $T \leq t$. However, by appending the indicator function of this event we obtain a Markov chain denoted,

$$\Psi^*(t) = (\Psi(t), \mathbf{1}\{T \leq t\}),$$

with state space $\mathsf{X}^* = \mathsf{X} \times \mathsf{X} \times \{0, 1\}$. The subset $\mathsf{X} \times \mathsf{X} \times \{1\}$ is absorbing for this chain.

The following two propositions allow us to infer properties of $\boldsymbol{\Psi}^*$ based on properties of $\boldsymbol{X}$. The proof of Proposition A.5.1 is immediate from the definitions.

Proposition A.5.1. *Suppose that $\boldsymbol{X}$ satisfies (V3) with f coercive. Then (V3) holds for the bivariate chain $\boldsymbol{\Psi}^*$ in the form,*

$$\mathsf{E}[V_*(\Psi(t+1)) \mid \Psi(t) = (x, y)^{\mathsf{T}}] \leq V_*(x, y) - f_*(x, y) + b_*,$$

with $V_*(x, y) = V(x) + V(y)$, $f_*(x, y) = f(x) + f(y)$, and $b_* = 2b$. Consequently, there exists $b_0 < \infty$ such that,

$$\mathsf{E}\Bigl[\sum_{t=0}^{T-1} \bigl(f(X(t)) + f(Y(t))\bigr)\Bigr] \leq 2[V(x) + V(y)] + b_0, \qquad x, y \in \mathsf{X}.$$

□

A necessary condition for the Mean Ergodic Theorem for arbitrary initial conditions is aperiodicity. Similarly, aperiodicity is both necessary and sufficient for x^{**}-irreducibility of $\mathbf{\Psi}^*$ with $x^{**} := (x^*, x^*, 1)^{\rm T} \in \mathsf{X}^*$:

Proposition A.5.2. *Suppose that $\boldsymbol{X}$ is x^*-irreducible and aperiodic. Then the bivariate chain is x^{**}-irreducible and aperiodic.*

Proof. Fix any $x, y \in \mathsf{X}$, and define

$$n_0 = \min\{n \ge 0 : P^n(x, x^*)P^n(y, x^*) > 0\}.$$

The minimum is finite since $\boldsymbol{X}$ is x^*-irreducible and aperiodic. We have $\mathsf{P}\{T \le n\} = 0$ for $n < n_0$ and by the construction of $\mathbf{\Psi}$,

$$\mathsf{P}\{T = n_0\} = \mathsf{P}\{\Psi(n_0) = (x^*, x^*)^{\rm T} \mid T \ge n_0\} = P^{n_0}(x, x^*)P^{n_0}(y, x^*) > 0.$$

This establishes x^{**}-irreducibility.

For $n \ge n_0$ we have,

$$\mathsf{P}\{\Psi^*(n) = x^{**}\} \ge \mathsf{P}\{T = n_0,\ \Psi^*(n) = x^{**}\} = P^{n_0}(x, x^*)P^{n_0}(y, x^*)P^{n-n_0}(x^*, x^*).$$

The right hand side is positive for all $n \ge 0$ sufficiently large since $\boldsymbol{X}$ is aperiodic. □

A.5.2 Mean ergodic theorem

A mean ergodic theorem is obtained based upon the following *coupling inequality*:

Proposition A.5.3. *For any given $g\colon \mathsf{X} \to \mathbb{R}$ we have,*

$$\big|\mathsf{E}[g(X(t))] - \mathsf{E}[g(Y(t))]\big| \le \mathsf{E}[(|g(X(t))| + |g(Y(t))|)\mathbf{1}(T > t)].$$

If $Y(0) \sim \pi$ so that $\boldsymbol{Y}$ is stationary we thus obtain,

$$|\mathsf{E}[g(X(t))] - \pi(g)| \le \mathsf{E}[(|g(X(t))| + |g(Y(t))|)\mathbf{1}(T > t)].$$

Proof. The difference $g(X(t)) - g(Y(t))$ is *zero* for $t \ge T$. □

The f*-total variation norm* of a signed measure μ on X is defined by

$$\|\mu\|_f = \sup\{|\mu(g)| : \|g\|_f \le 1\}.$$

When $f \equiv 1$ then this is exactly twice the *total-variation norm*: For any two probability measures π, μ,

$$\|\mu - \pi\|_{tv} := \sup_{A \subset \mathsf{X}} |\mu(A) - \pi(A)|.$$

Theorem A.5.4. *Suppose that $\boldsymbol{X}$ is aperiodic, and that the assumptions of Theorem A.4.5 hold. Then,*

(i) $\|P^t(x,\,\cdot\,) - \pi\|_f \to 0$ *as* $t \to \infty$ *for each* $x \in \mathsf{X}$.

(ii) *There exists* $b_0 < \infty$ *such that for each* $x, y \in \mathsf{X}$,

$$\sum_{t=0}^{\infty} \|P^t(x,\,\cdot\,) - P^t(y,\,\cdot\,)\|_f \leq 2[V(x) + V(y)] + b_0.$$

(iii) *If in addition* $\pi(V) < \infty$, *then there exists* $b_1 < \infty$ *such that*

$$\sum_{t=0}^{\infty} \|P^t(x,\,\cdot\,) - \pi\|_f \leq 2V(x) + b_1.$$

The coupling inequality is only useful if we can obtain a bound on the expectation $\mathsf{E}[|g(X(t))|\mathbf{1}(T > t)]$. The following result shows that this vanishes when $\boldsymbol{X}$ and $\boldsymbol{Y}$ are each stationary.

Lemma A.5.5. *Suppose that* $\boldsymbol{X}$ *is aperiodic, and that the assumptions of Theorem A.4.5 hold. Assume moreover that* $X(0)$ *and* $Y(0)$ *each have distribution* π, *and that* $\pi(|g|) < \infty$. *Then,*

$$\lim_{t\to\infty} \mathsf{E}[(|g(X(t))| + |g(Y(t))|)\mathbf{1}(T > t)] = 0.$$

Proof. Suppose that $\boldsymbol{X}$, $\boldsymbol{Y}$ are defined on the two-sided time-interval with marginal distribution π. It is assumed that these processes are independent on $\{0, -1, -2, \dots\}$. By stationarity we can write,

$$\begin{aligned}\mathsf{E}_\pi[|g(X(t))|\mathbf{1}(T > t)] &= \mathsf{E}_\pi[|g(X(t))|\mathbf{1}\{\Psi(i) \neq (x^*, x^*)^{\mathsf{T}},\ i = 0, \dots, t\}] \\ &= \mathsf{E}_\pi[|g(X(0))|\mathbf{1}\{\Psi(i) \neq (x^*, x^*)^{\mathsf{T}},\ i = 0, -1, \dots, -t\}]\,.\end{aligned}$$

The expression within the expectation on the right hand side vanishes as $t \to \infty$ with probability one by $(x^*, x^*)^{\mathsf{T}}$-irreducibility of the stationary process $\{\Psi(-t) : t \in \mathbb{Z}_+\}$. The Dominated Convergence Theorem then implies that

$$\lim_{t\to\infty} \mathsf{E}[|g(X(t))|\mathbf{1}(T > t)] = \mathsf{E}_\pi[|g(X(0))|\mathbf{1}\{\Psi(i) \neq (x^*, x^*)^{\mathsf{T}},\ i = 0, -1, \dots, -t\}] = 0.$$

Repeating the same steps with $\boldsymbol{X}$ replaced by $\boldsymbol{Y}$ we obtain the analogous limit by symmetry. ☐

Proof of Theorem A.5.4. We first prove (ii). From the coupling inequality we have, with $X(0) = x$, $X^\circ(0) = y$,

$$\begin{aligned}|P^t g\,(x) - P^t g\,(y)| &= |\mathsf{E}[g(X(t))] - \mathsf{E}[g(Y(t))]| \\ &\leq \mathsf{E}\big[\big(|g(X(t))| + |g(Y(t))|\big)\mathbf{1}(T > t)\big] \\ &\leq \|g\|_f \mathsf{E}\big[\big(f(X(t)) + f(Y(t))\big)\mathbf{1}(T > t)\big]\end{aligned}$$

Taking the supremum over all g satisfying $\|g\|_f \le 1$ then gives,

$$\|P^t(x,\,\cdot\,) - P^t(y,\,\cdot\,)\|_f \le \mathsf{E}\big[\big(f(X(t)) + f(Y(t))\big)\mathbf{1}(T > t)\big], \tag{A.54}$$

so that on summing over t,

$$\begin{aligned}\sum_{t=0}^{\infty} \|P^t(x,\,\cdot\,) - P^t(y,\,\cdot\,)\|_f &\le \sum_{t=0}^{\infty} \mathsf{E}\big[\big(f(X(t)) + f(Y(t))\big)\mathbf{1}(T > t)\big] \\ &= \mathsf{E}\Big[\sum_{t=0}^{T-1}\big(f(X(t)) + f(Y(t))\big)\Big].\end{aligned}$$

Applying Proposition A.5.1 completes the proof of (ii).

To see (iii) observe that,

$$\sum_{y\in\mathsf{X}} \pi(y)|P^t g\,(x) - P^t g\,(y)| \ge \Big|\sum_{y\in\mathsf{X}} \pi(y)[P^t g\,(x) - P^t g\,(y)]\Big| = |P^t g\,(x) - \pi(g)|.$$

Hence by (ii) we obtain (iii) with $b_1 = b_0 + 2\pi(V)$.

Finally we prove (i). Note that we only need establish the mean ergodic theorem in (i) for a single initial condition $x_0 \in \mathsf{X}$. To see this, first note that we have the triangle inequality,

$$\|P^t(x,\,\cdot\,)-\pi(\,\cdot\,)\|_f \le \|P^t(x,\,\cdot\,)-P^t(x_0,\,\cdot\,)\|_f + \|P^t(x_0,\,\cdot\,)-\pi(\,\cdot\,)\|_f, \qquad x, x_0 \in \mathsf{X}.$$

From this bound and Part (ii) we obtain,

$$\limsup_{t\to\infty} \|P^t(x,\,\cdot\,) - \pi(\,\cdot\,)\|_f \le \limsup_{t\to\infty} \|P^t(x_0,\,\cdot\,) - \pi(\,\cdot\,)\|_f.$$

Exactly as in (A.54) we have, with $X(0) = x_0$ and $Y(0) \sim \pi$,

$$\|P^t(x_0,\,\cdot\,) - \pi(\,\cdot\,)\|_f \le \mathsf{E}\big[\big(f(X(t)) + f(Y(t))\big)\mathbf{1}(T > t)\big]. \tag{A.55}$$

We are left to show that the right hand side converges to zero for some x_0. Applying Lemma A.5.5 we obtain,

$$\lim_{t\to\infty} \sum_{x,y} \pi(x)\pi(y)\mathsf{E}\big[[f(X(t)) + f(Y(t))]\mathbf{1}(T > t) \mid X(0) = x,\ Y(0) = y\big] = 0.$$

It follows that the right hand side of (A.55) vanishes as $t \to \infty$ when $X(0) = x_0$ and $Y(0) \sim \pi$. $\square$

A.5.3 Geometric ergodicity

Theorem A.5.4 provides a mean ergodic theorem based on the coupling time T. If we can control the tails of the coupling time T then we obtain a rate of convergence of $P^t(x,\,\cdot\,)$ to π.

The chain is called *geometrically recurrent* if $\mathsf{E}_{x^*}[\exp(\varepsilon\tau_{x^*})] < \infty$ for some $\varepsilon > 0$. For such chains it is shown in Theorem A.5.6 that for a.e. $[\pi]$ initial condition $x \in \mathsf{X}$, the total variation norm vanishes geometrically fast.

Theorem A.5.6. *The following are equivalent for an aperiodic, x^*-irreducible Markov chain:*

(i) *The chain is geometrically recurrent.*

(ii) *There exists $V : \mathsf{X} \to [1, \infty]$ with $V(x_0) < \infty$ for some $x_0 \in \mathsf{X}$, $\varepsilon > 0$, $b < \infty$, and a finite set $S \subset \mathsf{X}$ such that*

$$\mathcal{D}V(x) \le -\varepsilon V(x) + b\mathbf{1}_S(x), \qquad x \in \mathsf{X}. \tag{\textbf{V4}}$$

(iii) *For some $r > 1$,*

$$\sum_{n=0}^{\infty} \|P^n(x^*, \cdot) - \pi(\cdot)\|_1 r^n < \infty.$$

If any of the above conditions hold, then with V given in (ii), we can find $r_0 > 1$ and $b < \infty$ such that the stronger mean ergodic theorem holds: For each $x \in \mathsf{X}$, $t \in \mathbb{Z}_+$,

$$\|P^t(x, \cdot) - \pi(\cdot)\|_V := \sup_{|g| \le V} \big| \mathsf{E}_x[g(X(t)) - \pi(t)] \big| \le b r_0^{-t} V(x). \tag{A.56}$$

□

In applications Theorem A.5.6 is typically applied by constructing a solution to the drift inequality (V4) to deduce the ergodic theorem in (A.56). The following result shows that (V4) is not that much stronger than Foster's criterion.

Proposition A.5.7. *Suppose that the Markov chain $\boldsymbol{X}$ satisfies the following three conditions:*

(i) *There exists $V : \mathsf{X} \to (0, \infty)$, a finite set $S \subset \mathsf{X}$, and $b < \infty$ such that Foster's Criterion (V2) holds.*

(ii) *The function V is uniformly Lipschitz,*

$$l_V := \sup\{|V(x) - V(y)| : x, y \in \mathsf{X},\ \|x - y\| \le 1\} < \infty.$$

(iii) *For some $\beta_0 > 0$, $b_1 < \infty$,*

$$b_1 := \sup_{x \in \mathsf{X}} \mathsf{E}_x[e^{\beta_0 \|X(1) - X(0)\|}] < \infty.$$

Then, there exists $\varepsilon > 0$ such that the controlled process is V_ε-uniformly ergodic with $V_\varepsilon = \exp(\varepsilon V)$.

Proof. Let $\widetilde{\Delta}_V = V(X(1)) - V(X(0))$, so that $\mathsf{E}_x[\widetilde{\Delta}_V] \le -1 + b\mathbf{1}_S(x)$ under (V2). Using a second order Taylor expansion we obtain for each x and $\varepsilon > 0$,

$$
\begin{aligned}
[V_\varepsilon(x)]^{-1} P V_\varepsilon(x) &= \mathsf{E}_x\big[\exp\big(\varepsilon \widetilde{\Delta}_V\big)\big] \\
&= \mathsf{E}_x\big[1 + \varepsilon \widetilde{\Delta}_V + \tfrac{1}{2}\varepsilon^2 \widetilde{\Delta}_V^2 \exp\big(\varepsilon \vartheta_x \widetilde{\Delta}_V\big)\big] \qquad \text{(A.57)} \\
&\le 1 + \varepsilon\big(-1 + b\mathbf{1}_S(x)\big) + \tfrac{1}{2}\varepsilon^2 \mathsf{E}_x\big[\widetilde{\Delta}_V^2 \exp\big(\varepsilon \vartheta_x \widetilde{\Delta}_V\big)\big]
\end{aligned}
$$

where $\vartheta_x \in [0,1]$. Applying the assumed Lipschitz bound and the bound $\frac{1}{2}z^2 \le e^z$ for $z \ge 0$ we obtain, for any $a > 0$,

$$
\begin{aligned}
\tfrac{1}{2}\widetilde{\Delta}_V^2 \exp\big(\varepsilon \vartheta_x \widetilde{\Delta}_V\big) &\le a^{-2} \exp\big((a+\varepsilon)\big|\widetilde{\Delta}_V\big|\big) \\
&\le a^{-2} \exp\big((a+\varepsilon) l_V \big\|X(1) - X(0)\big\|\big)
\end{aligned}
$$

Setting $a = \varepsilon^{1/3}$ and restricting $\varepsilon > 0$ so that $(a+\varepsilon) l_V \le \beta_0$, the bound (A.57) and (iii) then give,

$$
[V_\varepsilon(x)]^{-1} P V_\varepsilon(x) \le (1-\varepsilon) + \varepsilon b \mathbf{1}_S(x) + \varepsilon^{4/3} b_1
$$

This proves the theorem, since we have $1 - \varepsilon + \varepsilon^{4/3} b_1 < 1$ for sufficiently small $\varepsilon > 0$, and thus (V4) holds for V_ε. □

A.5.4 Sample paths and limit theorems

We conclude this section with a look at the sample path behavior of partial sums,

$$
S_g(n) := \sum_{t=0}^{n-1} g(X(t)) \qquad \text{(A.58)}
$$

We focus on two limit theorems under (V3):

LLN The *Strong Law of Large Numbers* holds for a function g if for each initial condition,

$$
\lim_{n\to\infty} \frac{1}{n} S_g(n) = \pi(g) \qquad \text{a.s..} \qquad \text{(A.59)}
$$

CLT The *Central Limit Theorem* holds for g if there exists a constant $0 < \sigma_g^2 < \infty$ such that for each initial condition $x \in \mathsf{X}$,

$$
\lim_{n\to\infty} \mathsf{P}_x\Big\{(n\sigma_g^2)^{-1/2} S_{\tilde g}(n) \le t\Big\} = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx
$$

where $\tilde g = g - \pi(g)$. That is, as $n \to \infty$,

$$
(n\sigma_g^2)^{-1/2} S_{\tilde g}(n) \xrightarrow{\text{w}} N(0,1).
$$

The LLN is a simple consequence of the coupling techniques already used to prove the mean ergodic theorem when the chain is aperiodic and satisfies (V3). A slightly different form of coupling can be used when the chain is periodic. There is only room for a survey of theory surrounding the CLT, which is most elegantly approached using martingale methods. A relatively complete treatement may be found in [367], and the more recent survey [282].

The following versions of the LLN and CLT are based on Theorem 17.0.1 of [367].

Theorem A.5.8. *Suppose that $\boldsymbol{X}$ is positive Harris recurrent and that the function g satisfies $\pi(|g|) < \infty$. Then the LLN holds for this function.*

If moreover (V4) holds with $g^2 \in L_\infty^V$ then,

(i) *Letting $\tilde{g}$ denote the centered function $\tilde{g} = g - \int g \, d\pi$, the constant*

$$\sigma_g^2 := \mathsf{E}_\pi[\tilde{g}^2(X(0))] + 2\sum_{t=1}^{\infty} \mathsf{E}_\pi[\tilde{g}(X(0))\tilde{g}(X(t))] \tag{A.60}$$

is well defined, non-negative and finite, and

$$\lim_{n\to\infty} \frac{1}{n} \mathsf{E}_\pi\big[\big(S_{\tilde{g}}(n)\big)^2\big] = \sigma_g^2. \tag{A.61}$$

(ii) *If $\sigma_g^2 = 0$ then for each initial condition,*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} S_{\tilde{g}}(n) = 0 \qquad \text{a.s.}.$$

(iii) *If $\sigma_g^2 > 0$ then the CLT holds for the function g.*

□

The proof of the theorem in [367] is based on consideration of the martingale,

$$M_g(t) := \hat{g}(X(t)) - \hat{g}(X(0)) + \sum_{i=0}^{t-1} \tilde{g}(X(i)), \qquad t \ge 1,$$

with $M_g(0) := 0$. This is a martingale since Poisson's equation $P\hat{g} = \hat{g} - \tilde{g}$ gives,

$$\mathsf{E}[\hat{g}(X(t)) \mid X(0), \dots, X(t-1)] = \hat{g}(X(t-1)) - \tilde{g}(X(t-1)),$$

so that,

$$\mathsf{E}[M_g(t) \mid X(0), \dots, X(t-1)] = M_g(t-1).$$

The proof of the CLT is based on the representation $S_{\tilde{g}}(t) = M_g(t) + \hat{g}(X(t)) - \hat{g}(X(0))$, combined with limit theory for martingales, and the bounds on solutions to Poisson's equation given in Theorem A.4.5.

An alternate representation for the asymptotic variance can be obtained through the alternate representation for the martingale as the partial sums of a martingale difference sequence,

$$M_g(t) = \sum_{i=1}^{t} \widetilde{\Delta}_g(i), \qquad t \ge 1,$$

with $\{\widetilde{\Delta}_g(t) := \hat{g}(X(t)) - \hat{g}(X(t-1)) + \tilde{g}(X(t-1))\}$. Based on the martingale difference property,

$$\mathsf{E}[\widetilde{\Delta}_g(t) \mid \mathcal{F}_{t-1}] = 0, \qquad t \ge 1,$$

it follows that these random variables are uncorrelated, so that the variance of $\boldsymbol{M}_g$ can be expressed as the sum,

$$\mathsf{E}[(M_g(t))^2] = \sum_{i=1}^{t} \mathsf{E}[(\widetilde{\Delta}_g(i))^2], \qquad t \ge 1.$$

In this way it can be shown that the asymptotic variance is expressed as the steady-state variance of $\widetilde{\Delta}_g(i)$. For a proof of (A.62) (under conditions much weaker than assumed in Proposition A.5.9) see [367, Theorem 17.5.3].

Proposition A.5.9. *Under the assumptions of Theorem A.5.8 the asymptotic variance can be expressed,*

$$\sigma_g^2 = \mathsf{E}_\pi[(\widetilde{\Delta}_g(0))^2] = \pi(\hat{g}^2 - (P\hat{g})^2) = \pi(2g\hat{g} - g^2). \tag{A.62}$$

□

A.6 Converse theorems

The aim of Section A.5 was to explore the application of (V3) and the coupling method. We now explain why (V3) is *necessary* as well as sufficient for these ergodic theorems to hold.

Converse theorems abound in the stability theory of Markov chains. Theorem A.6.1 contains one such result: If $\pi(f) < \infty$ then there is a solution to (V3), defined as a certain "value function". For a x^*-irreducible chain the solution takes the form,

$$PV_f = V_f - f + b_f \mathbf{1}_{x^*}, \tag{A.63}$$

where the Lyapunov function V_f defined in (A.64) is interpreted as the 'cost to reach the state x^*'. The identity (A.63) is a dynamic programming equation for the *shortest path problem* described in Section 9.4.1.

Theorem A.6.1. *Suppose that $\boldsymbol{X}$ is a x^*-irreducible, positive recurrent Markov chain on X and that $\pi(f) < \infty$, where $f\colon \mathsf{X} \to [1,\infty]$ is given. Then, with*

$$V_f(x) := \mathsf{E}_x\Big[\sum_{t=0}^{\sigma_{x^*}} f(X(t))\Big], \qquad x \in \mathsf{X}, \tag{A.64}$$

the following conclusions hold:

(i) *The set $\mathsf{X}_f = \{x : V_f(x) < \infty\}$ is non-empty and absorbing:*

$$P(x, \mathsf{X}_f) = 1 \quad \textit{for all } x \in \mathsf{X}_f.$$

(ii) *The identity (A.63) holds with* $b_f := \mathsf{E}_{x^*}\Big[\sum_{t=1}^{\tau_{x^*}} f(X(t))\Big] < \infty.$

(iii) *For* $x \in \mathsf{X}_f$,

$$\lim_{t\to\infty} \frac{1}{t}\mathsf{E}_x[V_f(X(t))] = \lim_{t\to\infty} \mathsf{E}_x[V_f(X(t))\mathbf{1}\{\tau_{x^*} > t\}] = 0.$$

Proof. Applying the Markov property, we obtain for each $x \in \mathsf{X}$,

$$\begin{aligned} PV_f(x) &= \mathsf{E}_x\Big[\mathsf{E}_{X(1)}\Big[\sum_{t=0}^{\sigma_{x^*}} f(X(t))\Big]\Big] \\ &= \mathsf{E}_x\Big[\mathsf{E}\Big[\sum_{t=1}^{\tau_{x^*}} f(X(t)) \mid X(0), X(1)\Big]\Big] \\ &= \mathsf{E}_x\Big[\sum_{t=1}^{\tau_{x^*}} f(X(t))\Big] = \mathsf{E}_x\Big[\sum_{t=0}^{\tau_{x^*}} f(X(t))\Big] - f(x), \qquad x \in \mathsf{X}. \end{aligned}$$

On noting that $\sigma_{x^*} = \tau_{x^*}$ for $x \neq x^*$, the identity above implies the desired identity in (ii).

Based on (ii) it follows that X_f is absorbing. It is non-empty since it contains x^*, which proves (i).

To prove the first limit in (iii) we iterate the idenitity in (ii) to obtain,

$$\mathsf{E}_x[V_f(X(t))] = P^t V_f\,(x) = V_f(x) + \sum_{k=0}^{t-1}[-P^k f\,(x) + b_f P^k(x, x^*)], \quad t \geq 1.$$

Dividing by t and letting $t \to \infty$ we obtain, whenever $V_f(x) < \infty$,

$$\lim_{t\to\infty} \frac{1}{t}\mathsf{E}_x[V_f(X(t))] = \lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1}[-P^k f\,(x) + b_f P^k(x, x^*)].$$

Applying (i) and (ii) we conclude that the chain can be restricted to X_f, and the restricted process satisfies (V3). Consequently, the conclusions of the Mean Ergodic Theorem A.5.4 hold for initial conditions $x \in \mathsf{X}_f$, which gives

$$\lim_{t\to\infty} \frac{1}{t}\mathsf{E}_x[V_f(X(t))] = -\pi(f) + b_f \pi(x^*),$$

and the right hand side is zero for by (ii).

By the definition of V_f and the Markov property we have for each $m \geq 1$,

$$\begin{aligned} V_f(X(m)) &= \mathsf{E}_{X(m)}\Big[\sum_{t=0}^{\sigma_{x^*}} f(X(t))\Big] \\ &= \mathsf{E}\Big[\sum_{t=m}^{\tau_{x^*}} f(X(t)) \mid \mathcal{F}_m\Big], \quad \text{on } \{\tau_{x^*} \geq m\}. \end{aligned} \tag{A.65}$$

Moreover, the event $\{\tau_{x^*} \geq m\}$ is $\mathcal{F}_m$ measurable. That is, one can determine if $X(t) = x^*$ for some $t \in \{1, \dots, m\}$ based on $\mathcal{F}_m := \sigma\{X(t) : t \leq m\}$. Consequently, by the smoothing property of the conditional expectation,

$$\mathsf{E}_x[V_f(X(m))\mathbf{1}\{\tau_{x^*} \geq m\}] = \mathsf{E}\Big[\mathbf{1}\{\tau_{x^*} \geq m\}\mathsf{E}\Big[\sum_{t=m}^{\tau_{x^*}} f(X(t)) \mid \mathcal{F}_m\Big]\Big]$$

$$= \mathsf{E}\Big[\mathbf{1}\{\tau_{x^*} \geq m\}\sum_{t=m}^{\tau_{x^*}} f(X(t))\Big] \leq \mathsf{E}\Big[\sum_{t=m}^{\tau_{x^*}} f(X(t))\Big]$$

If $V_f(x) < \infty$, then the right hand side vanishes as $m \to \infty$ by the Dominated Convergence Theorem. This proves the second limit in (iii). □

Proposition A.6.2. *Suppose that the assumptions of Theorem A.6.1 hold: $\boldsymbol{X}$ is a x^*-irreducible, positive recurrent Markov chain on X with $\pi(f) < \infty$. Suppose that there exists $g \in L_\infty^f$ and $h \in L_\infty^{V_f}$ satisfying,*

$$Ph = h - g.$$

Then $\pi(g) = 0$, so that h is a solution to Poisson's equation with forcing function g. Moreover, for $x \in \mathsf{X}_f$,

$$h(x) - h(x^*) = \mathsf{E}_x\Big[\sum_{t=0}^{\tau_{x^*}-1} g(X(t))\Big]. \qquad \text{(A.66)}$$

Proof. Let $M_h(t) = h(X(t)) - h(X(0)) + \sum_{k=0}^{t-1} g(X(k))$, $t \geq 1$, $M_h(0) = 0$. Then $\boldsymbol{M}_h$ is a zero-mean martingale,

$$\mathsf{E}[M_h(t)] = 0, \quad \textit{and} \quad \mathsf{E}[M_h(t+1) \mid \mathcal{F}_t] = M_h(t), \qquad t \geq 0.$$

It follows that the stopped process is a martingale,

$$\mathsf{E}[M_h(\tau_{x^*} \wedge (r+1)) \mid \mathcal{F}_r] = M_h(\tau_{x^*} \wedge r), \qquad r \geq 0.$$

Consequently, for any r,

$$0 = \mathsf{E}_x[M_h(\tau_{x^*} \wedge r)] = \mathsf{E}_x\Big[h(X(\tau_{x^*} \wedge r)) - h(X(0)) + \sum_{t=0}^{\tau_{x^*} \wedge r-1} g(X(t))\Big].$$

On rearranging terms and subtracting $h(x^*)$ from both sides,

$$h(x) - h(x^*) = \mathsf{E}_x\Big[[h(X(r)) - h(x^*)]\mathbf{1}\{\tau_{x^*} > r\} + \sum_{t=0}^{\tau_{x^*} \wedge r-1} g(X(t))\Big], \qquad \text{(A.67)}$$

where we have used the fact that $h(X(\tau_{x^*} \wedge t)) = h(x^*)$ on $\{\tau_{x^*} \leq t\}$.

Applying Theorem A.6.1 (iii) and the assumption that $h \in L_\infty^{V_f}$ gives,

$$\limsup_{r\to\infty}\Big|\mathsf{E}_x\big[\big(h(X(r))-h(x^*)\big)\mathbf{1}\{\tau_{x^*}>r\}\big]\Big|$$
$$\le (\|h\|_{V_f}+|h(x^*)|)\limsup_{r\to\infty}\mathsf{E}_x[V_f(X(r))\mathbf{1}\{\tau_{x^*}>r\}]=0.$$

Hence by (A.67), for any $x\in\mathsf{X}_f$,

$$h(x)-h(x^*)=\lim_{r\to\infty}\mathsf{E}_x\Big[\sum_{t=0}^{\tau_{x^*}\wedge r-1} g(X(t))\Big].$$

Exchanging the limit and expectation completes the proof. This exchange is justified by the Dominated Convergence Theorem whenever $V_f(x)<\infty$ since $g\in L_\infty^f$. □