Reliability by Design in Distributed Power Transmission Networks

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Abstract

The system operator of a large power transmission network must ensure that power is delivered whenever there is demand in order to maintain highly reliable electric service. To fulfill this mandate, the system operator must procure reserve capacity to respond to unforeseen events such as an unexpected surge in demand. This paper constructs a centralized optimal solution for a power network model by generalizing recent techniques for the centralized optimal control of demand-driven production systems. The optimal solution indicates how reserves must be adjusted according to environmental factors including variability, and the ramping-rate constraints on generation. Sensitivity to transmission constraints is addressed through the construction of an effective cost on an aggregate model.

Key words: Optimization; inventory theory; networks; power transmission.

1 Introduction

The system operator of a large power transmission network must ensure that power is delivered *reliably* whenever there is demand. This is a non-trivial task in a complex network subject to unexpected surges in demand, unplanned generator outages, or transmission line failures. To hedge against this uncertainty, the system operator must procure reserve capacity in excess of forecast demand.

These issues surrounding the effective management of a power grid are analogous to the dynamic version of the *newsboy problem*: a news-vendor must decide how many papers to purchase from a wholesaler in the face of uncertain demand. Any extra papers will have no value at the end of the day, but of course sales will be missed if newspapers are not purchased in sufficient quantities [19,35].

Because power failure is extremely costly, as witnessed in the Northeast black-outs in 2003, the power consumer is willing to pay substantial amounts of money in return for improving reliability of service. This is the reason the California Independent System Operator (ISO) continually monitors reserves, and updates on an hourly basis predictions of generation capacity and demand, as illustrated in Figure 1.

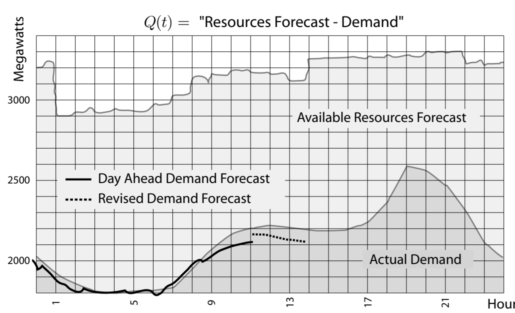

Fig. 1. Estimated demand and total on-line capacity in the California power network, on March 27, 2005. This plot is based on data obtained from the California Independent System Operator website.

There are several factors that have hindered the rigorous investigation of reliability in power networks:

(i) *A realistic model of a power transmission network*

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is typically highly complex.

(ii) *Transmission lines and generation ramping rates are subject to strict limits.*

(iii) *Power networks are subject to dynamic and unpredictable events, such as weather or demand surges.*

Although the dynamic nature of the wholesale electricity market is well recognized, the existing literature is largely built around static market models [2,7], which appear inadequate to examine various economic issues arising from the dynamic nature of the market. The first attempt to build a manageable dynamic model to investigate these economic issues appeared in [11], where the *single-consumer* dynamic model described in (17) was first introduced. This is a stochastic model of a single generator supplying power to a single consumer whose (normalized) demand evolves according to a driftless Brownian motion. One can view this model as an example of *resource pooling*: If the transmission lines are not congested, different generators with similar characteristics can be aggregated to form a single generator, and different utilities can be similarly aggregated.

The structure of the optimal solution is identified in the general model subject to transmission constraints. The optimal policy is based on the construction of an effective cost on an aggregate model. Consequently, its form depends on transmission constraints, as well as variability of demand, and the ramping-rate constraints on generation.

The remainder of this paper is organized as follows. Section 2 contains a review of optimization techniques for inventory models, new extensions to flow models, and performance characterization for the single-consumer power network. A dynamic network model is proposed in Section 3 that explicitly addresses finite transmission line limits and other network structure. Much of the analysis is based on an *aggregate relaxation*, which allows for a drastic reduction in system dimension, while maintaining most of the structure required for policy synthesis. Conclusions are contained in Section 4.

2 Network Optimization

In this section we describe models of inventory systems and power transmission networks. The models are similar, and consequently so are the respective optimal policies.

The control synthesis approach advanced in this paper is based on these three steps:

1. Construct a fluid model based on first-order statistics. A lower dimensional relaxation is constructed based on this deterministic model, and an effective policy is obtained for the relaxation based on an associated *effective cost*.
2. Introduce variability in the relaxation, and investigate its impact on optimal or heuristic control solutions.
3. Translate the control solution obtained in (2.) back to the original (unrelaxed) stochastic network.

The present paper will focus on the first two steps. Translation is the subject of several recent papers (see [29] for a survey.)

2.1 Inventory Systems

We begin in a deterministic setting in which the network is described using the *fluid model*. The fluid model can be interpreted as the mean-flow description of the network, or a time-scaled approximation of a stochastic model [13].

The vector of queue-lengths $\boldsymbol{q}$ in the fluid model evolves in a polyhedral region $\mathsf{X} \in \mathbb{R}^\ell$ representing buffer constraints. The fluid model can be conveniently represented as a differential inclusion $\frac{d^+}{dt}q \in \mathsf{V}$, where the velocity space $\mathsf{V} \subset \mathbb{R}^\ell$ is also a polyhedron, of the form

$$\mathsf{V} = \{v \in \mathbb{R}^\ell : \langle \xi^r, v\rangle \geq -(o_r - \rho_r), \quad 1 \leq r \leq \ell_v\}, \tag{1}$$

where the constants $\{o_r\}$ take on the values zero or one, $\alpha \in \mathbb{R}^\ell$ denotes the vector of exogenous demand and arrivals to the network, $\rho_r := \langle \xi^r, \alpha\rangle$, and $\xi^r \in \mathbb{R}^\ell$ for each $r = 1, \ldots, \ell_v$ [28].

The vector $\xi^r \in \mathbb{R}^\ell$ is called a *workload vector* if $o_r = 1$. It is assumed that the vectors are ordered so that $\{\xi^r : 1 \leq r \leq \ell_r\}$ are the ℓ_r workload vectors. The constants $\{\rho_r : 1 \leq r \leq \ell_r\}$ are called *load-parameters*; the vector load is given by $\rho = (\rho_1, \ldots, \rho_{\ell_r})^T$; and the maximum $\rho_\bullet$ is called the *network load*. The rth component of the ℓ_r-dimensional *workload process* is defined by $w_r(t) = \langle \xi^r, q(t)\rangle$, $1 \leq r \leq \ell_r$, $t \geq 0$.

Under mild conditions on the model [28], the network is stabilizable if and only if $\rho_\bullet < 1$, and in this case the minimal time to reach the origin can be expressed,

$$T^*(x) = \max_{1\leq r\leq \ell_v} \frac{\langle \xi^r, x\rangle}{o_r - \rho_r}, \quad x \in \mathsf{X}. \tag{2}$$

Network policies are typically based on a given cost function $c\colon \mathsf{X} \to \mathbb{R}_+$. The *infinite-horizon cost* is defined as,

$$J_*(x) = \inf\Big(\int_0^\infty c(q(t))dt\Big), \quad q(0) = x \in \mathsf{X}, \tag{3}$$

where the infimum is over all policies.

A stochastic model with queue-length process $\boldsymbol{Q}$ can be constructed to address the impact of variability. It will be useful to recall the following standard optimality criteria for a stochastic model: The optimal *average-cost* is given by,

$$\phi^*(x) := \inf \limsup_{T\to\infty} \mathsf{E}_x\Big[\frac{1}{T}\int_0^T c(Q(s))\,ds\Big], \qquad (4)$$

and the optimal *discounted-cost* is,

$$K^*(x;\eta) := \inf \int_0^\infty e^{-\eta s}\mathsf{E}_x[c(Q(s))]\,ds, \qquad (5)$$

where $\eta > 0$ is the discount parameter, and $Q(0) = x \in \mathsf{X}$ is the initial condition. The discounted cost K^* is also called the discounted value function. In both (4) and (5), the infimum is over all policies.

In a series of papers it is shown that an appropriate translation of an infinite-horizon optimal policy for the fluid model is approximately optimal for certain classes of stochastic models, or that the policies share important structural properties [21,27,1,10,28,29].

The strongest results are obtained in a 'heavy-traffic' setting in which $\rho_\bullet \approx 1$. For the deterministic model one can construct a policy for $\boldsymbol{q}$ that is very nearly optimal based on an n-dimensional *workload relaxation* [28, Theorem 4.1]. Analogous results hold for a stochastic model under significantly more restrictive assumptions (see [1,28,29], and related results and examples in [24,26,4,25,10,9,30,8].)

The construction of a workload relaxation with state process $\widehat{\boldsymbol{q}}$ is based on the geometry of the velocity space V. Suppose that the workload vectors are ordered so that the load parameters $\{\rho_r\}$ are decreasing in r. For $n \le \ell_r$, the nth workload relaxation is defined as the differential inclusion $\frac{d^+}{dt}\widehat{q} \in \widehat{\mathsf{V}}$, where the (possibly unbounded) velocity space $\widehat{\mathsf{V}}$ is given by the intersection of n half-spaces,

$$\widehat{\mathsf{V}} = \{v : \langle \xi^r, v\rangle \ge -(1-\rho_r), 1 \le r \le n\}. \qquad (6)$$

The n-dimensional workload process is defined by $\widehat{w}(t) = \widehat{\Xi}\widehat{q}(t)$, $t \ge 0$, where the $n \times \ell$ *workload matrix* is given by $\widehat{\Xi} := [\xi^1 \mid \cdots \mid \xi^n]^T$.

The workload process $\widehat{\boldsymbol{w}}$ obeys the dynamics,

$$\widehat{w}(t) = w - (1-\rho)t + \iota(t), \quad t \ge 0, \quad \widehat{w}(0) = w, \qquad (7)$$

where the idleness process $\boldsymbol{\iota}$ is non-decreasing, and $w := \widehat{\Xi}q(0)$ is the initial condition. The idleness process is interpreted as a control process for $\widehat{\boldsymbol{w}}$. The workload process is constrained to the *workload space*, defined as the n-dimensional polyhedron,

$$\widehat{\mathsf{W}} := \{\widehat{\Xi}x : x \in \mathsf{X}\}.$$

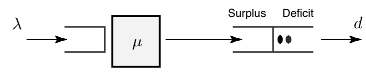

Fig. 2. Simple make-to-stock system.

A stochastic workload model is defined through the introduction of a Brownian disturbance $\boldsymbol{N}$: The *controlled Brownian-motion (CBM) model* is defined as follows,

$$\widehat{W}(t) = w - (1-\rho)t + I(t) + N(t), \quad t \ge 0. \qquad (8)$$

The state process $\widehat{\boldsymbol{W}}$, the idleness process $\boldsymbol{I}$, and the disturbance process $\boldsymbol{N}$ are subject to the following constraints: (i) $\widehat{W}(t) \in \widehat{\mathsf{W}}$; (ii) $\boldsymbol{N}$ is a drift-less Brownian motion, with instantaneous covariance matrix Σ; and (iii) The idleness process $\boldsymbol{I}$ is the *control process for the workload process*. It is assumed to be non-decreasing, and adapted to the Brownian motion $\boldsymbol{N}$.

An optimal control problem for $\widehat{\boldsymbol{w}}$ or $\widehat{\boldsymbol{W}}$ is defined using the *effective cost* $\bar{c}$: $\widehat{W} \to \mathbb{R}_+$, whose definition is based on the given cost function c: $\mathsf{X} \to \mathbb{R}_+$ for the ℓ-dimensional queueing model. For a given $w \in \widehat{\mathsf{W}}$, it is expressed as the solution to the nonlinear program,

$$\bar{c}(w) = \mathbf{min} \;\; c(x) \quad \mathbf{s.t.} \quad \widehat{\Xi}x = w, \quad x \in \mathsf{X}. \qquad (9)$$

The control criteria considered for a workload model based on the effective cost are entirely analogous to those defined for $\boldsymbol{q}$ or $\boldsymbol{Q}$.

Consider the simple make-to-stock model shown in Figure 2, consisting of a single processing station with rate μ, controlled release of raw material into the system with maximum rate denoted λ, and exogenous demand with rate d. A virtual queue is used to model inventory: a positive value indicates deficit (unsatisfied demand), and a negative value indicates excess inventory. The cost function c: $\mathsf{X} \to \mathbb{R}_+$ is assumed to be piecewise linear, $c(x) = c_1x_1 + c^+x_{2+} + c^-x_{2-}$, where $x_+ := \mathbb{I}\{x > 0\}x$ and $x_- := \mathbb{I}\{x < 0\}|x|$.

A two-dimensional fluid model is obtained on setting $\mathsf{X} = \mathbb{R}_+ \times \mathbb{R}$, and the velocity space $\mathsf{V} \subset \mathbb{R}^2$ the parallelogram defined by (1) with $\xi^1 = -\xi^3 = \mu^{-1}(0,1)$ and $\xi^2 = -\xi^4 = \lambda^{-1}(-1,1)$. The vector load is $\rho = d(\mu^{-1}, \lambda^{-1})^T$.

Suppose that $\lambda \gg \mu > d$. In this case a one-dimensional workload relaxation of the form (7) or (8) can be justified based on the first workload vector, and the effective cost is given by $\bar{c}(w) = c^+\mu w_+ + c^-\mu w_-$ for $w \in \mathsf{W} := \mathbb{R}$.

Optimization of the fluid model is trivial: a pathwise optimal solution drives the workload process to the origin in minimal time. The optimal policy for the CBM model under the average-cost criterion is obtained in [36]. The solution is a threshold policy, defined so that $\widehat{\boldsymbol{W}}$ is a one-dimensional reflected Brownian motion (RBM) on $[-\beta, \infty)$ for some threshold $\beta \in \mathbb{R}$. In either the discounted or the average-cost problem, the threshold is given by

$$\beta^* = \frac{1}{\gamma} \ln\Big(1 + \frac{c^+}{c^-}\Big), \qquad (10)$$

where the constant $\gamma > 0$ depends upon the specific performance criterion.

Theorem 2.1 extends the main result of [36] to the discounted cost criterion.

Theorem 2.1 *The following hold for the simple make-to-stock CBM model:*

(i) *The discounted-cost optimal policy is the threshold policy with parameter (10), where $\gamma(\eta) > 0$ is the positive root of the quadratic equation*

$$\tfrac{1}{2}\sigma^2\gamma^2 - \delta\gamma - \eta = 0\,, \qquad (11)$$

with $\delta = 1 - d/\mu$, and σ^2 the instantaneous variance of $\boldsymbol{N}$.

(ii) *The average-cost optimal policy is also a threshold policy, whose threshold is the limiting value $\gamma = \lim_{\eta\downarrow 0} \gamma(\eta) = 2\delta/\sigma^2$.*

Proof: For (ii) see [36].

The discounted cost can be expressed

$$\widehat{K}(w;\eta) = \eta^{-1}\mathsf{E}[\bar{c}(\widehat{W}(\tau)) \mid \widehat{W}(0) = w]\,, \qquad (12)$$

where τ be an exponential random variable with parameter η that is independent of $\boldsymbol{N}$. Consider a threshold policy such that $\widehat{\boldsymbol{W}}$ is an RBM on $[-\beta, \infty)$, and fix an initial condition $\widehat{W}(0) = w < -\beta$, so that $\widehat{W}(0+) = -\beta$. In this case the random variable $\widehat{W}(\tau)$ has an exponential distribution, with parameter $\gamma(\eta)$ (this follows from identities in [3, p. 250].) The formula for $\beta^*(\eta)$ is obtained on differentiating the expectation (12) with respect to β, and setting the derivative equal to zero. □

The true value of workload relaxation lies in the modeling and control of complex networks.

The results that follow will be illustrated using the network shown at left in Figure 3. Shown in the figure are 14-queues at 5-stations. Two of the queues are virtual, representing excess and backlog inventories of finished products. Routing and/or scheduling decisions must be made at each station. This network is examined in detail in [9].

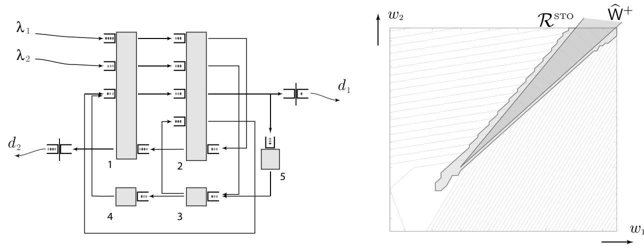

Fig. 3. The figure on the left shows a 14-buffer network. The figure on the right shows its effective cost and optimal fluid and stochastic policies.

For the general workload model, the *monotone region* is given by,

$$\widehat{\mathsf{W}}^+ := \left\{ \begin{array}{l} w \in \widehat{\mathsf{W}} \textit{ such that } \bar{c}(w) \le \bar{c}(w') \\ \textit{whenever } w' \ge w, \textit{ and } w' \in \widehat{\mathsf{W}} \end{array} \right\},$$

and the effective cost is called *monotone* if $\widehat{\mathsf{W}}^+ = \widehat{\mathsf{W}}$.

For small values of n, an optimal policy can often be found through visual inspection. The following result from [10, Proposition 3.2] makes this precise.

Proposition 2.2 *Suppose that c is linear. Then, for the fluid workload relaxation (7),*

(i) *If $n = 1$ then there exists a unique pathwise optimal solution from each initial condition.*

(ii) *If $n = 2$ then a pathwise optimal solution exists from each initial condition if, and only if, the the two dimensional vector $1 - \rho$ lies within the monotone region $\widehat{\mathsf{W}}^+$. The optimal solution satisfies $\widehat{w}^*(t) \in \widehat{\mathsf{W}}^+$ for all $t > 0$, and any initial condition.* □

The 14-buffer network satisfies the assumptions of Proposition 2.2 (ii) under mild assumptions on the processing rates. For the specific numerical values of demand and processing rates considered in [9], a two-dimensional workload relaxation can be justified. Suppose that the cost function c is linear, with marginal cost $(1, 5, 10)$ for respectively internal inventory, excess inventory at a virtual buffer, and unsatisfied demand. Contour plots of the resulting two-dimensional effective cost calculated using (9) are shown at right in Figure 3.

If $1-\rho$ lies within the monotone region $\widehat{\mathsf{W}}^+$ then the optimal policy for the fluid model is defined by the monotone region. The two boundaries of $\widehat{\mathsf{W}}^+$ are interpreted as switching curves for the fluid model $\widehat{\boldsymbol{w}}$.

Consider now the two-dimensional stochastic model (8). It is shown in [10] that the optimal policy for $\widehat{\boldsymbol{W}}$ is approximated by an affine translation of the optimal policy for the fluid model. A refinement of this result is presented in Theorem 2.3 below.

An optimal policy can again be defined by a pair of switching curves. In the discussion that follows we restrict to the discounted-cost control criterion. The lower switching curve that defines $\widehat{\boldsymbol{I}}_2^*$ can be expressed

$$s_*^{\rm sto}(w_1;\eta) = \inf(w_2 \ge 0 : \frac{\partial \widehat{K}^*}{\partial w_2}(w;\eta) > 0), \qquad (13)$$

where $\widehat{K}^*$ is the discounted value function. An analogous representation holds for the upper switching curve. For simplicity we focus attention here on the lower switching curve that defines $\boldsymbol{I}_2^*$.

The proof of Theorem 2.3 is based on consideration of the *height process*, defined as the solution to the stochastic differential equation (SDE),

$$dH(t) = -\delta_H dt + I_H(t) + dN_H, \quad t \ge 0, \qquad (14)$$

where $\boldsymbol{N}_H$ is a one-dimensional standard Brownian motion, and the drift $\delta_H \in \mathbb{R}$ and the instantaneous variance $\sigma_H^2 \ge 0$ for $\boldsymbol{N}_H$ are given by

$$\delta_H = \frac{\langle \bar{c}^+ - \bar{c}^-, 1-\rho\rangle}{\langle \bar{c}^+ - \bar{c}^-, e^2\rangle}, \quad \sigma_H^2 = \frac{(\bar{c}^+ - \bar{c}^-)^T \Sigma (\bar{c}^+ - \bar{c}^-)}{\langle \bar{c}^+ - \bar{c}^-, e^2\rangle^2}.$$

The allocation process $\boldsymbol{I}_H$ is non-decreasing and adapted to $\boldsymbol{N}_H$. The height process is a relaxation of the vertical projection, $\frac{\langle \bar{c}^+ - \bar{c}^-, \widehat{W}(t)\rangle}{\langle \bar{c}^+ - \bar{c}^-, e^2\rangle}$. A sample path of the height process for a particular example is shown later in Figure 8.

A cost function $\bar{c}_H : \mathbb{R} \to \mathbb{R}_+$ for the height process is defined by,

$$\bar{c}_H(h) = c^+ h_+ + c^- h_-\,, \qquad (15)$$

where $c^+ = \bar{c}_2^+$, and $c^- = |\bar{c}_2^-|$.

Theorem 2.3 *Consider the CBM model (8) with dimension $n = 2$, and let $s_*^{\rm sto}$ denote the lower switching curve defined in (13) that defines the idleness process $\boldsymbol{I}_2$ for the discounted-cost-optimal control problem with discount factor $\eta > 0$. Then,*

$$s_*^{\rm fluid}(w_1) - s^{\rm sto}(w_1;\eta) \to \beta^*(\eta), \quad w_1 \to \infty, \qquad (16)$$

where the optimal affine-shift parameter is given by (10) with $c^+ = \bar{c}_2^+$, $c^- = |\bar{c}_2^-|$, and $\gamma(\eta)$ the positive solution to the quadratic equation (11) with $\sigma = \sigma_H, \delta = \delta_H$.

Proof: The existence of asymptote defined by $\beta^*(\eta)$ is established using the height process in [10, Proposition 4.2, Theorem 4.4]. The proof of these results is based upon optimization of the discounted cost for $\boldsymbol{H}$ with the cost function defined in (15). Optimization of the height process to obtain the desired expression for $\beta^*(\eta)$ is identical to the proof of Theorem 2.1 (i). □

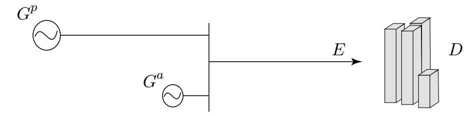

Fig. 4. A simple power network with two-suppliers, and a single consumer

To illustrate these results, consider again the network shown in Figure 3. A discrete-time *controlled random walk* model of the form (8) was considered in which all processes were restricted to the integer lattice, and $\boldsymbol{N}$ was a random walk - details may be found in [9]. The average-cost optimal policy is defined by a region $\mathcal{R}^{\rm sto} \subset \mathbb{Z}^2$ that determines the optimal idleness process $\boldsymbol{I}^*$ as a function of $\widehat{\boldsymbol{W}}$. Shown at right in Figure 3 is the optimal region obtained using value iteration. Note that the boundaries of $\mathcal{R}^{\rm sto}$ are approximately affine, and they are approximately consistent with the formula (10).

2.2 Single Consumer Power Network

Here we focus on the simple power network illustrated in Figure 4, in which a single utility is connected to a primary and an ancillary generator through a transmission line with infinite capacity.

Throughout this paper, it is assumed that there are two broad classes of generators. The *primary generators* deliver the bulk of the power, while *ancillary generators* are smaller, more responsive generators with correspondingly higher operational cost. We also implicitly assume that the mean demand for power is met by the primary services, which is scheduled in advance. The residual demand process, in which the mean demand is subtracted, is denoted $\boldsymbol{D} = \{D(t) : t \ge 0\}$. With respect to the data shown in Figure 1, we can take $D(t)$ equal to the difference between the day-ahead forecast and the actual demand at time t over a 24-hour period.

The *on-line capacity* (or, simply *capacity*) of a generator is defined as the maximum amount of power that can be extracted from the generator instantaneously.[1] The respective capacities of the primary and ancillary generators at time t are denoted $G^p(t)$ and $G^a(t)$. Capacity is subject to strict ramp-rate limits: For constants ζ^{p+}, ζ^{a+}, and for each $0 \le t_0 < t_1 < \infty$,

$$\frac{G^p(t_1) - G^p(t_0)}{t_1 - t_0} \le \zeta^{p+}, \quad \frac{G^a(t_1) - G^a(t_0)}{t_1 - t_0} \le \zeta^{a+}\,.$$

We have relaxed *lower* rate constraints on the generators, and transmission line constraints will be ignored in the analysis of this section.

[1] On-line capacity is different from the name plate capacity, which represents the technical production capacity specified by the manufacturer of the generator.

The relaxation of the lower rate constraints makes it possible to construct a tractable model analogous to the simple make-to-stock model. Suppose that the demand process $\boldsymbol{D}$ is a zero-mean Brownian motion with instantaneous variance $\sigma_D^2 > 0$, and denote the reserve at time t by,

$$Q(t) = G^p(t) + G^a(t) - D(t), \quad t \geq 0\,. \tag{17}$$

We say that *black-out* occurs at time t if $Q(t) < 0$. We use the term black-out interchangeably with the failure of reliable services.

We impose the following assumptions to simplify the characterization of the optimal solution:

Constant marginal production cost for primary and ancillary service: (c^p, c^a) is the pair of the marginal costs for the primary and the ancillary services.

Constant marginal disutility of black-out: If there is excess demand for power, then black-out occurs. The social cost of excess demand is given by $c^{bo}|q|$ when $q < 0$, where $c^{bo} > 0$. The larger the shortage, the greater the damage to society.

The cost function on (G^p, G^a, Q) thus has the following form,

$$\mathcal{C}(t) := c^p G^p(t) + c^a G^a(t) + c^{bo} Q_-(t), \quad t \geq 0\,. \tag{18}$$

It is assumed that $c^{bo} > c^a > c^p$. Ancillary service is more expensive to run than primary service, but the marginal social cost of black-out is far more costly. We assume $\zeta^{a+} > \zeta^{p+}$. The higher cost of ancillary service is justified by its faster ramp-up rate.

For the purpose of optimization we consider the two dimensional controlled diffusion $\boldsymbol{X} := (\boldsymbol{Q}, \boldsymbol{G}^a)^T$, which can be equivalently expressed for an initial condition $X(0) = x$ via,

$$X(t) = x + B\zeta^+ t - BI(t) - D_X(t), \quad t \geq 0, \tag{19}$$

where $\zeta^+ = (\zeta^{p+}, \zeta^{a+})^T$, $D_X(t) = (D(t), 0)^T$, and B is the 2×2 matrix

$$B = \begin{bmatrix} 1, & 1 \\ 0, & 1 \end{bmatrix} \tag{20}$$

The state process $\boldsymbol{X}$ is controlled through the two-dimensional allocation process $\boldsymbol{I}$, which is assumed to be non-decreasing and adapted to $\boldsymbol{D}_X$.

The state process evolves on the state space $\mathsf{X} := \mathbb{R} \times \mathbb{R}_+$, and the cost function (18) is reduced to a cost function on X as follows: For $x = (q, g^a)^T \in \mathsf{X}$ define,

$$c(x) := c^p q + (c^a - c^p) g^a + c^{bo} q_-\,. \tag{21}$$

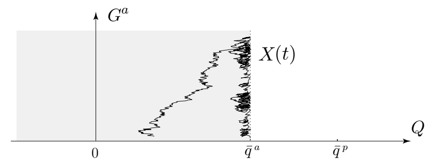

Fig. 5. Trajectory of $\boldsymbol{X}$ under an affine policy

The cost function (21) represents the cost (18) in the sense that $\mathsf{E}[\mathcal{C}(t)] = \mathsf{E}[c(X(t))]$.

The average-cost optimal policy for $\boldsymbol{X}$ is obtained in [12]. It is precisely an affine policy, defined by two thresholds that determine idleness of $G^p(t)$ and $G^a(t)$. For a given pair of thresholds $\bar{q} = (\bar{q}^p, \bar{q}^a)^T$, the affine policy is defined so that $\boldsymbol{X}$ is a reflected Brownian motion within the two-dimensional region,

$$\mathsf{X}_{\bar{q}} := \{x_1 \leq \bar{q}^a,\ x_2 \geq 0\} \cup \{x_1 \leq \bar{q}^p,\ x_2 = 0\}. \tag{22}$$

A sketch of a typical sample path of $\boldsymbol{X}$ under an affine policy is shown in Figure 5.

The following result, taken from [12], can be viewed as a refinement of Theorem 2.3, and the affine parameters are identical in form to the parameter β^* defined in (10). The proof is based on a detailed examination of the dynamic programming equations for this singular control problem [17,26,5].

Theorem 2.4 *Consider the two-dimensional diffusion model (19). The optimal policy is affine under either the discounted or average-cost criterion. In particular, the average-cost optimal policy is affine with parameters,*

$$\begin{aligned} \bar{q}^{a*} &= \tfrac{1}{\gamma_0} \ln\bigl(1 + \tfrac{c^{bo} - c^a}{c^a}\bigr), & \gamma_0 &= 2\tfrac{\zeta^{p+} + \zeta^{a+}}{\sigma_D^2} \\ \bar{q}^{p*} &= \bar{q}^{a*} + \tfrac{1}{\gamma_1} \ln\bigl(1 + \tfrac{c^a - c^p}{c^p}\bigr), & \gamma_1 &= 2\tfrac{\zeta^{p+}}{\sigma_D^2}. \end{aligned} \tag{23}$$

□

It is remarkable that the optimal policy is computable for this multi-dimensional model. Moreover, it is shown below in Theorem 2.5 that one can compute the optimal steady-state cost under any given affine policy.

Theorem 2.5 *For any affine policy, the Markov process $\boldsymbol{X}$ is exponentially ergodic [15]. The unique stationary distribution π on X satisfies,*

(i) *The first marginal of π is $\mathsf{P}_\pi\{Q(t) \leq q\} =$*

$$\begin{cases} e^{-\gamma_1(\bar{q}^p - q)} & \bar{q}^a \leq q \leq \bar{q}^p \\ e^{-\gamma_1(\bar{q}^p - \bar{q}^a) - \gamma_0(\bar{q}^a - q)} & q \leq \bar{q}^a. \end{cases} \tag{24}$$

(ii) *The steady state cost is $\phi(\bar{q}) =$*

$$\gamma_0^{-1}\Bigl(\frac{\zeta^{a+}}{\zeta^{p+}} c^a + e^{-\gamma_0 \bar{q}^a} c^{bo}\Bigr) e^{-\gamma_1(\bar{q}^p - \bar{q}^a)} + (\bar{q}^p - \gamma_1^{-1}) c^p.$$

Proof: The proof is based upon structure developed in Section A.1.

In particular, in the proof of Proposition A.2 it is shown that the function $V_2 \colon \mathsf{X} \to \mathbb{R}_+$ defined by $V_2(x) := \frac{1}{2}(q - g^a - \bar{q}^p)^2$, $x \in \mathsf{X}$, satisfies the following *Lyapunov drift condition*: For finite positive constants ε_2, b_2, and a compact set S,

$$\mathcal{D}V_2\,(x) \leq -\varepsilon_2 c(x) + b_2 \mathbb{I}_S(x), \quad x \in \mathcal{R}(\bar{q}), \tag{25}$$

where the differential generator is defined in (A.5).

Consider the function $V := \exp\bigl(\sqrt{1 + \varepsilon V_2}\bigr)$ with $\varepsilon > 0$. Straightforward calculations then show that for $\varepsilon > 0$ sufficiently small, there exists $\varepsilon_e > 0$, $b_e < \infty$, and a compact set $S_e \subset \mathcal{R}(\bar{q})$ such that

$$\mathcal{D}V\,(x) \leq -\varepsilon_e V(x) + b_e \mathbb{I}_{S_e}(x), \quad x \in \mathcal{R}(\bar{q}).$$

The function V also satisfies the boundary conditions (A.6) required to ensure that $\mathcal{D}V$ coincides with $\mathcal{A}V$, where $\mathcal{A}$ denotes the *extended generator* defined in Section A.1. It then follows that V satisfies a version of the 'drift inequality' (V4) of [32,15]. Exponential ergodicity then follows from the main result of [15].

Part (i) is Lemma A.7, and (ii) follows on combining (i) with Lemmas A.4 and A.8 □

3 Power Networks

We now consider the question of optimal reserves for a power network with general topology.

3.1 Network Model

The power network model consists of ℓ_n nodes, denoted $\mathcal{N} = \{1, \ldots, \ell_n\}$. Located at each node are one or more of the following: Primary generation, ancillary generation, and exogenous demand. The nodes are connected together by a set of transmission lines $\mathcal{L} = \{1, \ldots, \ell_t\}$.

The following DC power flow model is commonly used to model power flow over a transmission network [6]. Let $P \in \mathbb{R}^{\ell_n}$ denote the vector of nodal power flows (injections from generation minus extractions), let $F \in \mathbb{R}^{\ell_t}$ denote the vector of transmission line power flows, and let $\theta \in \mathbb{R}^{\ell_n}$ denote the vector of nodal voltage angles. The DC power flow model consists of the following two linear equations relating P, F and θ in equilibrium,

$$P = B_n \theta\,, \quad F = B_l T \theta, \tag{26}$$

where B_n is the nodal susceptance matrix, and B_l is the diagonal matrix whose ith entry is equal to the susceptance of line i. The node-to-branch incidence matrix T captures the topology of the power network, and also defines a positive direction of flow: If line i connects nodes j and k, and power flow from node j to node k is regarded as positive, then the ith row of the matrix T is $(e^j - e^k)^T$, where e^i denotes the ith standard basis element in $\mathbb{R}^{\ell_n}$. It is known that the three matrices are related by the formula $B_n = T^T B_l T$.

A linear equation relating P and F is obtained from the DC power flow equations (26) by solving for θ in the first equation, and then substituting this value into the second DC power flow equation, giving

$$F = [B_l T B_n^\dagger] P, \tag{27}$$

where $B_n^\dagger$ is a pseudo-inverse. The matrix $\Delta := B_l T B_n^\dagger$ is called the *distribution factor matrix* (DFM), and Equation (27) is known as the *DFM equation*. We maintain the linear constraints in (27), but introduce a stochastic demand process $\boldsymbol{D}$, and impose rate constraints on generation capacities.

The primary and ancillary generation capacities at node $i \in \mathcal{N}$ and time $t \geq 0$ are denoted $G_i^p(t)$ and $G_i^a(t)$, respectively, and the demand at node i is denoted $D_i(t)$. If node i does not contain a primary generator, then $G_i^p(t) = 0$ for all $t \geq 0$. The same convention is used for the absence of an ancillary generator at any node. The utility at node i schedules a generation capacity $E_i(t)$ that can be extracted at time t.

The primary generation profile is defined as the vector,

$$G^p(t) = (G_1^p(t), \ldots, G_{\ell_n}^p(t))^T.$$

The ancillary generation profile $G^a(t)$, demand profile $D(t)$, and extraction profile $E(t)$ are defined analogously. The reserve is defined as the difference,

$$Q(t) = E(t) - D(t), \quad t \geq 0. \tag{28}$$

The event $Q_i(t) < 0$ is interpreted as black-out, or the failure of reliable service at node i.

The remaining assumptions on the dynamic network model are described as follows:

(i) Generation capacities are subject to strict bounds: For $i \in \mathcal{N}$ and $t \geq 0$,

$$G_i^p(t) \leq \bar{G}_i^p, \quad G_i^a(t) \leq \bar{G}_i^a, \tag{29}$$

where $\bar{G}^p = (\bar{G}_1^p, \ldots, \bar{G}_{\ell_n}^p)^T$, $\bar{G}^a = (\bar{G}_1^a, \ldots, \bar{G}_{\ell_n}^a)^T$, are fixed ℓ_n-dimensional vectors.

(ii) Generation capacity is *rate constrained*: For $i \in \mathcal{N}$ and $0 \leq t_0 < t_1 < \infty$,

$$\tfrac{G_i^p(t_1) - G_i^p(t_0)}{t_1 - t_0} \leq \zeta_i^{p+}, \quad \tfrac{G_i^a(t_1) - G_i^a(t_0)}{t_1 - t_0} \leq \zeta_i^{a+},$$

where $\{\zeta^{p+}, \zeta^{a+}\}$, are ℓ_n-dimensional vectors with non-negative entries. If a primary or ancillary generator is absent at a given node then the respective rate is set to zero. For example, if node i does not contain a primary generator, then $\zeta_i^{p+} = 0$.

It is assumed that the non-zero rates are independent of i: For some strictly positive constants $\zeta^{p+} < \zeta^{a+}$, $\zeta_i^{p+} = \zeta^{p+}$ whenever $\zeta_i^{p+} \neq 0$. Similarly, $\zeta_i^{a+} = \zeta^{a+}$ if $\zeta_i^{a+} \neq 0$.

(iii) Generation and extraction must be balanced:

$$\sum_{i \in \mathcal{N}} G_i^p(t) + G_i^a(t) = \sum_{i \in \mathcal{N}} E_i(t), \quad t \geq 0.$$

(iv) Power flows over the network are consistent with the DC power flow model: The linear relationship (27) holds, $F(t) = \Delta P(t)$ for $t \geq 0$, where Δ is the $\ell_t \times \ell_n$ DFM, $F_i(t)$ denotes the power flow on transmission line $i \in \mathcal{L}$, and the vector of nodal power flows is denoted

$$P(t) = G^p(t) + G^a(t) - E(t). \tag{30}$$

(v) Power transmission is subject to strict constraints: There is a constraint vector $f^+ = (f_1^+, \dots, f_{\ell_t}^+)^T$ such that the line flows must satisfy for each t,

$$F(t) = \Delta P(t) \in \mathsf{F} := \{x \in \mathbb{R}^{\ell_t} : -f^+ \leq x \leq f^+\}.$$

The network shown in Figure 6 provides an example of the general model in which there is a source of demand at each of three nodes. We refer to this as the *3-node network* [23,18]. A primary generator is located at node 1, and an ancillary generator is located at each of nodes 2 and 3. The transmission lines are all assumed to have equal susceptance, and the directions of positive power flow are as indicated by the arrows in the figure.

The vector F of power flow on the transmission lines, and the vector P of nodal power flows are, respectively,

$$F = (F_1, F_2, F_3)^T, \quad P = (G_1^p - E_1, G_2^a - E_2, G_3^a - E_3)^T,$$

and the DFM is given by,

$$\Delta = \frac{1}{3}\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix} \tag{31}$$

We return to this example in Section 3.4.

We maintain the notation introduced Section 2.2 in our consideration of the general power network. For each $i \in \mathcal{N}$, the primary and ancillary capacity processes at node i are expressed,

$$G_i^p(t) = \zeta^{p+}t - I_i^p(t), \quad G_i^a(t) = \zeta^{a+}t - I_i^a(t), \quad t \geq 0,$$

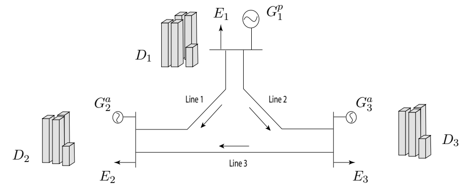

Fig. 6. Three-node power transmission network.

where $\boldsymbol{I}_i^p$ and $\boldsymbol{I}_i^a$, $t \geq 0$, are adapted to $\boldsymbol{D}$, and are non-decreasing. Eventually we will restrict to Gaussian demand, so that the process $\boldsymbol{D}$ is an ℓ_n-dimensional Brownian motion

To formulate a control problem we introduce a cost function in analogy to that constructed in Section 2.2: for given primary generation, ancillary generation, extraction, and demand profiles $\{g^p, g^a, e, d\} \subset \mathbb{R}^{\ell_n}$,

$$c(g^p, g^a, e, d) := \sum_{i \in \mathcal{N}} \bigl(c_i^p g_i^p + c_i^a g_i^a + c_i^{bo}(e_i - d_i)_-\bigr). \tag{32}$$

The vectors c^p, c^a, and c^{bo} of primary power cost, ancillary power cost, and cost of black-out are interpreted as previously. For example, c_i^p is the marginal cost of primary generation at node $i \in \mathcal{N}$. If node i does not contain a primary generator, then $c_i^p = 0$. The same convention is used for the absence of an ancillary generator at any node.

Observe that the cost is a function of the $4\ell_n$ variables contained in the ℓ_n-dimensional vectors $\{g^p, g^a, e, d\}$. For a network with 1000 generators in operation, such as the California system, this model is not suitable for mathematical analysis. In the next subsection we show how a form of aggregate relaxation leads to a tractable model for control design and analysis.

3.2 Complete Resource Pooling

Transmission line constraints complicate both modeling and optimization since one must consider the spatial distribution of demand and supply. The rates of change of power production are limited by laws of physics and operational safety requirements. However, 'scheduled power extraction' is not subject to stringent constraints. We therefore impose no direct rate constraints on the process $\boldsymbol{E}$ beyond what is already assumed based on the model assumptions described in Section 3.1.

For a given pair of generation profiles $\{g^p, g^a\}$, we denote by $\mathcal{K}(g^p, g^a)$ the set of *admissible extraction profiles*. That is, $e \in \mathcal{K}(g^p, g^a)$ if and only if,

$$\sum_{i \in \mathcal{N}} g_i^p + g_i^a = \sum_{i \in \mathcal{N}} e_i, \quad \textit{and} \quad f = \Delta p \in \mathsf{F},$$

where p is the vector of nodal power flows defined as in (30) by,

$$p = (g_1^p + g_1^a - e_1, \dots, g_{\ell_n}^p + g_{\ell_n}^a - e_{\ell_n})^T.$$

Two extraction profiles $e', e'' \in \mathbb{R}^{\ell_n}$ are called *exchangeable* if $e', e'' \in \mathcal{K}(g^p, g^a)$ for some $g^p \in \mathbb{R}^{\ell_n}$, $g^a \in \mathbb{R}_+^{\ell_n}$. Any pair of exchangeable extraction profiles can be interchanged instantaneously since we have imposed no rate constraints on $\boldsymbol{E}$. However, on optimizing we will conclude that the process $E(t)$ is continuous as a function of time since demand varies continuously.

Based on the exchangeable extraction profiles, we are led to consideration of the *aggregate process* $\boldsymbol{X}_A = (\boldsymbol{Q}_A, \boldsymbol{G}_A^a)$, where for each $t \ge 0$,

$$Q_A(t) = \sum_{i\in\mathcal{N}} Q_i(t)\,, \quad G_A^a(t) = \sum_{i\in\mathcal{N}} G_i^a(t).$$

This can be expressed in a form analogous to the single-consumer model (19):

$$X_A(t) = x_A + B\zeta_A^+ t - BI_A(t) - D_X(t), \quad t \ge 0, \quad (33)$$

where $X_A(0) = x_A$ is the initial condition, B is defined in (20), $D_A(t) = \sum_{i\in\mathcal{N}} D(t)$, and the aggregate idleness process is given by,

$$I_A(t) := \sum_{i\in\mathcal{N}} [I_i^p(t), I_i^a(t)]^T, \quad t \ge 0.$$

For a given aggregate state $x_A = (q_A, g_A^a)^T \in \mathsf{X}_A := \mathbb{R} \times \mathbb{R}_+$, and demand profile $d \in \mathbb{R}^{\ell_n}$, the effective cost is defined as the value of the linear program,

$$\begin{aligned} \bar{c}(x_A, d) = \ \mathbf{min} \quad & \sum_{i\in\mathcal{N}} (c_i^p g_i^p + c_i^a g_i^a + c_i^{bo} q_{i-}) \\ \mathbf{s.t.} \quad & q_A = \textstyle\sum_{i\in\mathcal{N}}(e_i - d_i) \\ & g_A^a = \textstyle\sum_{i\in\mathcal{N}} g_i^a \\ & 0 = \textstyle\sum_{i\in\mathcal{N}}(g_i^p + g_i^a - e_i) \\ & 0 = e - d - q, \quad f = \Delta p, \end{aligned} \quad (34)$$

with $f \in \mathsf{F}$, $e, q, g^p \in \mathbb{R}^{\ell_n}$, $g^a \in \mathbb{R}_+^{\ell_n}$. If capacity generation constraints exist as in (29) then the constraints in (34) must be augmented accordingly.

Consideration of the dual LP shows that the effective cost is a piecewise linear function of $\{x_A, d, f^+\}$, of the form

$$\bar{c}(x_A, d) = \max_{1\le i\le \ell_c} \left(\langle \bar{c}^i, x_A\rangle + \langle \bar{c}_d^i, d\rangle + \langle \bar{c}_f^i, f^+\rangle\right), \quad (35)$$

where $\ell_c \ge 1$ is equal to the number of possible optimizing extreme points in the dual. Each of the vectors $\{\bar{c}^i, \bar{c}_d^i, \bar{c}_f^i\}$ are interpreted as marginal effective cost vectors. In particular, if $i \in \{1, \dots, \ell_c\}$ is the unique maximizer in (35), then $\bar{c}^i \in \mathbb{R}^2$ is equal to sensitivity of cost with respect to the state variable x_A, $\bar{c}_d^i \in \mathbb{R}^{\ell_n}$ equals sensitivity with respect to demand, and $\bar{c}_f^i \in \mathbb{R}^{\ell_t}$ equals sensitivity with respect to transmission constraints.

Let $\mathsf{X} := \mathbb{R}^{\ell_n} \times \mathbb{R}_+^{\ell_n} \times \mathbb{R}^{\ell_n}$ denote the space of possible triples (g^p, g^a, e). The effective cost is obtained by choosing the 'cheapest' extraction profile that is consistent with the given demand, aggregate ancillary generation, and aggregate reserve. For a given demand profile $d \in \mathbb{R}^{\ell_n}$, and aggregated state $x_A = (q_A, g_A^a)^T \in \mathsf{X}_A$, the effective state $x^* = (g^{p*}, g^{a*}, e^*) = \mathcal{X}^*(x_A, d)$ is any optimizer in (34). That is,

$$\begin{aligned} \mathcal{X}^*(x_A, d) = \ & \mathop{\mathbf{arg\,min}}_{x\in\mathsf{X}} \ \bar{c}(x_A, d) \\ & \mathbf{s.t.} \ \sum_{i\in\mathcal{N}} (e_i - d_i) = q_A, \quad \sum_{i\in\mathcal{N}} g_i^a = g_A^a\,. \end{aligned} \quad (36)$$

The effective state is a sufficient statistic for control in the following sense: Given a policy for the two-dimensional relaxation, we obtain a policy for the complex network through the mapping $X^*(t) := \mathcal{X}^*(X_A^*(t), D(t))$. The generation and extraction profiles will give the lowest cost among all generation and extraction profiles that correspond to the aggregate state process $\boldsymbol{X}_A$. Feasibility requires continuity of this solution, which can be assumed without loss of generality:

Proposition 3.1 *The map $\mathcal{X}^*$ defined in (36) can be constructed so that it is continuous as a function from $\mathsf{X}_A \times \mathbb{R}^{\ell_n}$ to X.*

Proof: The result follows from [14, Theorem II.1.4 and Theorem I.3.2]. The conditions of the theorem hold for the LP defined in (36) since the feasible set has non-empty interior. □

The effective cost is a function of both the state and demand. This is essential since information contained in the demand profile is critical for effective control. Consider for example a network state $x \in \mathsf{X}$ and demand profile $d \in \mathbb{R}^{\ell_n}$ satisfying

$$d_A := \sum_{i\in\mathcal{N}} d_i = \sum_{i\in\mathcal{N}} g_i^p + g_i^a.$$

That is, the total amount of generation capacities equals the aggregate demand. Suppose there are no transmission constraints. Then, under general conditions $e = d$ in an optimal solution, which cannot be achieved based solely on knowledge of the aggregate demand d_A.

Under the equal-cost condition and in the absence of transmission constraints, the aggregate process (33) can be obtained from an aggregate relaxation of the general power network model presented in Section 3.1. This can be seen by the fact that under the setting of Proposition 3.2, the only distinction among any generators in the network is whether they are primary or ancillary generators.

Proposition 3.2 *Suppose that $f_i^+ = \infty$ for each i, so that the transmission lines are not constrained. Assume moreover that the two sets of cost parameters $\{c_i^p\}$ and $\{c_i^a\}$ are each independent of i. Then,*

(i) *The effective cost is the piecewise linear function of $x_A = (q_A, g_A^a)^T$ given by,*

$$\bar{c}(x_A, d) = c^p q_A + (c^a - c^p) g_A^a + c_{i^*}^{bo}(q_A)_- . \quad (37)$$

where $i^* := \arg\min_{i\in\mathcal{N}}\{c_i^{bo}\}$.

(ii) *Under either the discounted or average cost criterion, an optimal affine policy can be constructed based on the thresholds defined in Section 2.2. The optimal solution satisfies,*

$$(G^{p*}(t), G^{a*}(t), E^*(t)) = \mathcal{X}^*(X_A^*(t), D(t)) \quad t > 0,$$

and hence $Q_i^*(t) = 0$ for all $i \neq i^*$ and all t.

Proof: In the absence of transmission constraints the solution satisfying $Q_i^*(t) = 0$ for all $i \neq i^*$ and all $t > 0$ is feasible for the multidimensional network. Since this solution coincides with the optimal solution for the relaxation, it must also be optimal. □

3.3 Transmission Constraints

Here we develop the aggregate model in the presence of transmission line capacity constraints. We find that in certain asymptotic regimes the policy is approximated by an affine policy with computable parameters, even when f^+ is finite.

Since the effective cost defined in (34) is a function of both state and demand, for the purposes of optimization it is necessary to consider the Markov state process defined as the *pair* $(\boldsymbol{X}_A, \boldsymbol{D})$, and allow arbitrary initial conditions for demand. For a given initial condition $(x_A^0, d^0) \in \mathsf{X}_A \times \mathbb{R}^{\ell_n}$, the value function under the discounted cost criterion is defined

$$K(x_A^0, d^0) = \mathsf{E}_{x_A^0, d^0}\Big[\int_0^\infty e^{-\eta t}\bar{c}(X_A(t), D(t))\, dt\Big], \quad (38)$$

where η is the discount parameter.

To approximate the optimal policy we generalize the *second relaxation* of [10], leading to a version of the height process (14). It is assumed that for all states in some large connected region of interest, there are two dominant marginal effective cost vectors in the maximization (35), with indices denoted $i_1, i_2 \in \{1, \ldots, \ell_c\}$. That is, the value of $\langle \bar{c}^i, x_A\rangle + \langle \bar{c}_d^i, d\rangle + \langle \bar{c}_f^i, f^+\rangle$ is much smaller than $\bar{c}(x_A, d)$ for each $i \notin \{i_1, i_2\}$, and each state-demand pair (x_A, d) under consideration. For $x_A \in \mathsf{X}_A$, $d \in \mathbb{R}^{\ell_n}$, the relaxed effective cost is defined as the maximum,

$$\bar{\bar{c}}(x_A, d) = \max_{i=i_1,i_2} \left(\langle \bar{c}^i, x_A\rangle + \langle \bar{c}_d^i, d\rangle + \langle \bar{c}_f^i, f^+\rangle\right) . \quad (39)$$

It is assumed that the cost vectors $\{\bar{c}^{i_1}, \bar{c}^{i_2}\}$ are linearly independent, so that the maximization (39) is non-trivial. We obviously have $\bar{\bar{c}}(x_A, d) \le \bar{c}(x_A, d)$ for *all* pairs (x_A, d).

The discounted-cost optimal policy can be computed explicitly for the cost function $\bar{\bar{c}}$ if boundary constraints are ignored. The policy is affine, with a single threshold that can be constructed through consideration of the height process defined with respect to the monotone region for the cost function $\bar{\bar{c}}$. To perform this construction we transform the state equation into the form considered in Section 2.2.

Define the two dimensional process $Y(t) = MX_A(t) + M_d D(t) + M_f f^+$, $t \ge 0$, with

$$M = \left[\bar{c}^{i_1} \mid \bar{c}^{i_2}\right]^T,\ M_d = \left[\bar{c}_d^{i_1} \mid \bar{c}_d^{i_2}\right]^T,\ M_f = \left[\bar{c}_f^{i_1} \mid \bar{c}_f^{i_2}\right]^T . \quad (40)$$

The *workload process* is the two dimensional process obtained through a second linear transformation:

$$W(t) = -(MB)^{-1} Y(t), \quad t \ge 0, \quad (41)$$

where B is defined in (20). The time evolution of the workload process is expressed,

$$W(t) = w - \zeta_A^+ t + I_A(t) + D_W(t), \quad t \ge 0, \quad (42)$$

where $D_W(t) = B^{-1}(D_A(t) - M^{-1}M_d D(t))$. The idleness process $\boldsymbol{I}_A$ can be interpreted as a control process for $\boldsymbol{W}$.

For a given aggregated state $x_A \in \mathsf{X}_A$, demand $d \in \mathbb{R}^{\ell_n}$, and $y = Mx_A + M_d d + M_f f^+ \in \mathsf{Y}$, the relaxed effective cost is expressed as a function of y via $\bar{\bar{c}}(x_A, d) = \max(y_1, y_2)$. Consequently, we obtain the following formula for the relaxed effective cost as a function of w,

$$\bar{\bar{c}}(x_A, d) = \bar{\bar{c}}_W(w) := \max_{i=1,2}(-(MBw)_1, -(MBw)_2) . \quad (43)$$

The monotone region $\mathsf{W}^+ \subset \mathsf{W} := \mathbb{R}^2$ is defined to be the set of $w \in \mathbb{R}^2$ satisfying $\nabla \bar{\bar{c}}_W(w) \in \mathbb{R}_+^2$. Consideration of (43) shows that there are three possibilities: W^+ is a half space, $\mathsf{W}^+ = \mathbb{R}^2$, or $\mathsf{W}^+ = \emptyset$. The second and third cases are uninteresting: If $\mathsf{W}^+ = \mathbb{R}^2$ then the optimal policy sets $\boldsymbol{I}_A \equiv 0$, and if $\mathsf{W}^+ = \emptyset$ then $\boldsymbol{I}_A \equiv \infty$. If W^+ is a half space, then without loss of generality we assume that the product MB can be expressed,

$$-MB = [\bar{c}^+, \bar{c}^-]^T,$$

where $\bar{c}^+ \in \mathbb{R}^2_+$ is non-zero, and $\bar{c}^- \in \mathbb{R}^2$ with $\bar{c}^-_2 < 0$. Under these assumptions, the monotone region can be expressed,

$$\mathsf{W}^+ := \{w \in \mathbb{R}^2 : \langle \bar{c}^+, w\rangle \geq \langle \bar{c}^-, w\rangle\}.$$

The affine policy is defined to be an affine shift of the monotone region W^+. The affine shift is parameterized by a constant $\beta \in \mathbb{R}$, and the associated policy is defined so that $\boldsymbol{W}$ is a two dimensional reflected Brownian motion restricted to the region,

$$\mathcal{R}(\beta) := \{\mathsf{W}^+ - \beta e^2\} = \{w \in \mathbb{R}^2 : \langle \bar{c}^+ - \bar{c}^-, w + \beta e^2\rangle \geq 0\},$$

with reflection direction e^2 along the boundary of $\mathcal{R}(\beta)$.

The evaluation of a given value of $\beta > 0$ is performed using the following version of the height process described in Section 2,

$$H(t) = \frac{\langle \bar{c}^+ - \bar{c}^-, W(t)\rangle}{\langle \bar{c}^+ - \bar{c}^-, e^2\rangle}, \quad t \geq 0.$$

Under the affine policy $\boldsymbol{H}$ is the one-dimensional RBM on $[-\beta, \infty)$, whose drift vector and instantaneous variance are given by

$$\delta_H = \frac{\langle \bar{c}^+ - \bar{c}^-, \zeta^+_A\rangle}{\bar{c}^+_2 - \bar{c}^-_2}, \quad \sigma^2_H = \frac{(\bar{c}^+ - \bar{c}^-)^T \Sigma_W (\bar{c}^+ - \bar{c}^-)}{(\bar{c}^+_2 - \bar{c}^-_2)^2},$$

with Σ_W equal to the instantaneous covariance of $\boldsymbol{D}_W$. Recall that D_W is defined in terms of M and M_d, which are functions of the marginal effective cost vectors with respect to state and demand.

The optimal policy for the aggregated process $\boldsymbol{X}_A$ is approximated by this affine policy for large initial conditions. The proof of Proposition 3.3 is identical to the proof of [10, Theorem 4.4] (see Theorem 2.3 and surrounding discussion.)

Proposition 3.3 *For each $\eta > 0$ the optimal policy for $\boldsymbol{W}$ on $\mathbb{R}^2$ is defined by the affine policy with parameter given in (10), where $c^+ = \bar{c}^+_2$ and $c^- = |\bar{c}^-_2|$, so that $\boldsymbol{W}$ is an RBM on $\mathcal{R}(\beta^*(\eta))$.*

Consider a parameterized family of initial conditions $\{x^0_A(r), d(r) : r \geq 0\} \subset \mathsf{X}_A \times \mathbb{R}^{\ell_n}$ satisfying

$$\bar{c}(x_A(r), d(r)) = \langle \bar{c}^i, x_A\rangle + \langle \bar{c}^i_d, d\rangle + \langle \bar{c}^i_f, f^+\rangle, \quad i = i_1, i_2,$$

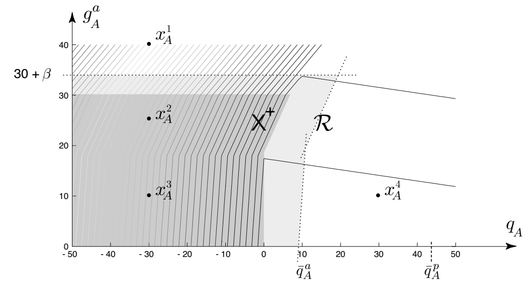

Fig. 7. Contour plot of the effective cost on X_A with demand fixed at $d = (-7, 9, -3)^T$. The dark grey region shows the monotone region of the effective cost function. This is equal to the intersection of all monotone regions obtained from each workload relaxation based on consideration of a pair of indices $\{i_1, i_2\}$ in (39). The light grey region represents an affine policy obtained from consideration of each of these workload process.

and suppose that $\|x_A(r)\| \to \infty$ as $r \to \infty$. Let $\boldsymbol{I}^*_A$ be the idleness process associated with the optimal policy for the discounted cost criterion (38), and let $\boldsymbol{I}^{w*}_A$ denote the idleness process associated with the constraint region $\mathcal{R}(\beta^*(\eta))$. Then,

$$\boldsymbol{I}^*_A \to \boldsymbol{I}^{w*}_A, \quad r \to \infty,$$

where the convergence is in distribution. □

We are not actually interested in 'large initial conditions' here. Rather, Proposition 3.3 provides intuition regarding the structure of an optimal policy. This result motivates the following heuristic to construct a policy for the power network: first construct a policy for $\boldsymbol{W}$, translate this from W to X_A, and then apply the map $\mathcal{X}^*$ to define the final policy for the original network.

3.4 The 3-Node Network

We return to the 3-node network shown in Figure 6 to illustrate Proposition 3.3 and the construction of a policy. The ramp-up rates are defined to be $\zeta^{p+} = 1$, $\zeta^{a+} = 3$, and the cost parameters are

$$c^p = (1, 0, 0)^T, \quad c^a = (0, 10, 10)^T. \tag{44}$$

The power lines are subject to the transmission constraints $F(t) \in \mathsf{F}$, where

$$\mathsf{F} = \{x \in \mathbb{R}^3 : -f^+ \leq x \leq f^+\}, \; f^+ = (13, 5, 8)^T.$$

For a given network state $x \in \mathsf{X}$ and demand profile d, the aggregated state $x_A = (q_A, g^a_A)^T \in \mathsf{X}_A$ is interpreted as follows,

$$q_A = \sum_{i \in \mathcal{N}} (e_i - d_i) = g^p_1 + g^a_2 + g^a_3 - \sum_{i=1}^{3} d_i, \quad g^a_A = g^a_1 + g^a_2.$$

For a given $x_A \in \mathsf{X}_A$ and demand vector $d \in \mathbb{R}^3$, the effective cost is the value of the LP,

$$
\begin{aligned}
\bar{c}(x_A, d) = \ \mathbf{min} \quad & c_1^p g_1^p + c_2^a g_2^a + c_3^a g_3^a + \sum_{i=\mathcal{N}} c_i^{bo} q_{i-} \\
\mathbf{s.t.} \quad & q_A = e_1 - d_1 + e_2 - d_2 + e_3 - d_3 \ , \\
& g_A^a = g_2^a + g_3^a \ , \\
& 0 = g_1^p + g_2^a + g_3^a - e_1 - e_2 - e_3 \ , \\
& 0 = e - d - q \ , \\
& f = \Delta p \ , \quad f \in \mathsf{F},
\end{aligned}
$$

where the DFM Δ is defined in (31). The solution of the dual gives a representation of the form (35), with $\ell_c = 4$ [8]. A contour plot of the effective cost is shown in Figure 7 with $d = (-7, 9, -3)^T$.

To illustrate the construction of the relaxed effective cost $\bar{\bar{c}}$ and the workload process $\boldsymbol{W}$, we first note that two sets of marginal effective cost parameters are dominant in a neighborhood of $x_A^0 = (-50, 50)^T$, $d^0 = (-7, 9, -3)^T$, and using (40) we obtain

$$M = \begin{bmatrix} -99, & 109 \\ -99, & 59 \end{bmatrix}, \ M_d = \begin{bmatrix} 1, & -99, & -99 \\ 1, & -49, & -99 \end{bmatrix}, \ M_f \approx \begin{bmatrix} 42, & 158, & 58 \\ 0, & -150, & 0 \end{bmatrix}$$

and for $t \geq 0$ the process $\boldsymbol{Y}$ defined above (40) is given by $Y(t) =$

$$M\big(x_A + B\zeta_A^+ t - BI_A(t) - \big(\begin{smallmatrix} D_A(t) \\ 0 \end{smallmatrix}\big)\big) + M_d D(t) + M_f f^+.$$

The workload process defined in (41) can be expressed in the form (42) with $\zeta_A^+ = (1, 6)^T$, and

$$W(0) = -\begin{bmatrix} 1 & ,-1 \\ 0 & , \ 1 \end{bmatrix} + \begin{bmatrix} -16 \\ 21 \end{bmatrix}, \quad D_W(t) = \tfrac{1}{99}\begin{bmatrix} 100 \ , & 10 \\ 0 \ , & 99 \end{bmatrix}\begin{pmatrix} D_1(t) \\ D_2(t) \end{pmatrix}.$$

The cost function on W is defined in (43) is,

$$\bar{\bar{c}}_W(w) = \max(99w_1 + 40w_2, 99w_1 - 10w_2), \quad w \in \mathsf{W}.$$

The height process $\boldsymbol{H}$ associated with the workload process $\boldsymbol{W}$ has an exponential distribution in steady state, with parameter $\gamma = 12/\sigma_{D_2}^2$, where $\sigma_{D_2}^2$ is the instantaneous variance of the demand process at node 2. The relaxed effective cost on W, a sample path of the workload process $\boldsymbol{W}$, and the associated height process $\boldsymbol{H}$ are shown in Figure 8. From (10) we obtain the following formula for the optimal parameter β^* when $\eta \to 0$:

$$\beta^* = \tfrac{1}{\gamma}\ln(5) = \tfrac{\ln(5)}{12}\sigma_{D_2}^2.$$

The policy for the workload model can be translated back to the aggregate process on X_A. The resulting policy

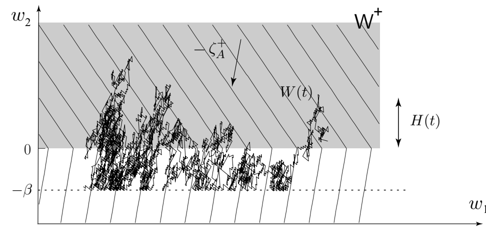

Fig. 8. The workload process $\boldsymbol{W}$ and the relaxed effective cost $\bar{\bar{c}}$ on the workload space W. Also illustrated is the associated height process, and the optimal affine policy which has parameter β.

defines a half space of the form,

$$\mathcal{R}_A = \{x_A \in \mathsf{X}_A : g_A^a \leq 30 + \beta^*\}.$$

Consideration of other cost parameters we obtain two other instances in which the effective cost on the resulting workload process possesses a monotone and non-monotone region. In each case we can construct a region in X_A similar to $\mathcal{R}_A$. If for each $x \in \mathsf{X}_A$ we consider the region obtained from the two dominant marginal effective cost vectors, we obtain the polyhedral region $\mathcal{R}$ shown in light grey in Figure 7. Within the interior of $\mathcal{R}$ we have $\frac{d}{dt}I_A(t) = 0$.

The dark grey region X^+ indicates the monotone region of the effective cost function $\bar{c}$: In a fluid model, for x_A within the interior of X^+ the cost is decreasing whenever each generator has non-negative ramp-rate. Each linear segment of the boundary of X^+ corresponds to a boundary of the monotone region in a workload relaxation based on consideration of a pair of indices $\{i_1, i_2\}$ in (39). Hence the construction of the set $\mathcal{R}$ that defines a policy amounts to constructing a polyhedral enlargement of the monotone region X^+, exactly as in the manufacturing example illustrated in Figure 3.

We stress that this policy is a function of both the state $X_A(t)$ and demand $D(t)$ since the effective cost is a function of both. In particular, $\mathcal{R} \subset \mathsf{X}_A$ varies with demand.

To fully understand this policy it is helpful to take a close look at the effective state. Consider first the aggregate state $x_A^1 = (-30, 40)^T$ shown in Figure 7. The effective state $\mathcal{X}^*(x_A^1)$ reveals that the reserve at node 1 is negative, which means that node 1 is experiencing black-out:

$$
\begin{array}{lll}
q_1 = -46.0 & q_2 = \ \ 3.06 & q_3 = \ \ 12.9 \\
g_1^p = -71.0 & g_2^a = \ \ 33.1 & g_3^a = \ \ 6.94 \\
e_1 = -53.0 & e_2 = \ \ 12.1, & e_3 = \ \ 9.94 \\
f_1 = -13.0 & f_2 = -5.00 & f_3 = -8.00
\end{array}
$$

It may be surprising to learn that ancillary service is ramping down at maximum rate. This is a consequence of the observation that each of the three transmission lines have reached their capacity limits: Nodes 2 and 3 cannot transfer reserves to node 1, and node 1 cannot extract more reserve to reduce the severity of black-out at node 1.

Blackout exists for the effective states $\mathcal{X}^*(x_A^2)$ and $\mathcal{X}^*(x_A^3)$, but in each of these cases there is sufficient transmission capacity to provide increased reserves to any node: Only line 2 is at its limit in the case of $\mathcal{X}^*(x_A^2)$, and for $\mathcal{X}^*(x_A^3)$ no transmission line is congested. The policy dictates that each generator ramp up at maximum rate, which is sensible under these conditions.

Finally, the effective state $\mathcal{X}^*(x_A^4, d)$ associated with x_A^4 corresponds to high power reserves at each node, so that generation ramps down.

4 Conclusion

This paper has introduced new network models to capture the complex topology and demand structure found in real power transmission networks. We have "abstracted away" some details in the actual operation of the transmission network such as the start-up cost of the generators and the cyclical component of the demand process to build analytically tractable models. Yet, the models considered here are significantly richer than static models, and at the same time, sufficiently tractable to reveal structure of optimal policies for power reserves in a dynamic setting. Extensions are addressed in [34,12], where in particular the centralized optimal outcome is used as a benchmark in evaluating candidate market designs.

Acknowledgements We are grateful to Profs. Peter Cramton, Tom Overbye, George Gross, Ramesh Johari, and Robert Wilson for helpful conversations, and Sebastian Neumayer at Illinois provided many suggestions. We also received significant feedback and inspiration at the IMA workshop, *Control and Pricing in Communication and Power Networks* [33]. We express our sincere thanks to Profs. Douglas Arnold, Chris DeMarco, Tom Kurtz, and Ruth Williams for creating this stimulating environment.

This paper is based upon work supported by the National Science Foundation under Award Nos. SES 00 04315 and ECS 02 17836. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

A Appendix

A.1 Affine Policies

Here we develop approaches to evaluating a given affine policy. It is valuable to have different approaches to analysis since some approaches provide precise conclusions, but are correspondingly more difficult to generalize to the general network setting.

First we establish the existence of a well-behaved solution to (17).

Proposition A.1 *For any $\bar{q}^a < \bar{q}^p < \infty$, the pair $\boldsymbol{X} = (\boldsymbol{Q}, \boldsymbol{G}^a)$ defined via (17) and the constraints above defines a strong Markov process on X.*

Proof: The sample paths of $\boldsymbol{X}$ can be constructed as a continuous, deterministic map from the sample paths of $\boldsymbol{D}$. To see this, consider first a deterministic demand process that is continuous, and piecewise linear. Then $\boldsymbol{X}$ can be constructed by induction on n, where n denotes the index of the successive maximal time-intervals on which $\boldsymbol{D}$ is linear.

Fix any time horizon $T > 0$, and view $\boldsymbol{D}$ and $\boldsymbol{X}$ as elements of respective vector spaces of continuous functions in the uniform topology $C^\infty[0, T]$. The map from $\boldsymbol{D}$ to $\boldsymbol{X}$ is uniformly continuous in this setting when $\boldsymbol{D}$ is piecewise linear. This completes the proof since by definition the sample paths of Brownian motion are continuous, and any continuous function can be approximated uniformly by piecewise linear functions on $[0, T]$. ☐

To compute the steady-state cost we compute a solution to *Poisson's equation*,

$$\mathcal{A}h = -c + \phi, \quad x \in \mathcal{R}(\bar{q}), \tag{A.1}$$

where $\phi = \phi(\bar{q})$ is the infinite horizon cost defined in (4), which is independent of $x \in \mathsf{X}$ when $\boldsymbol{X}$ is controlled using an affine policy. The solution h is assumed to be continuous on X, with quadratic growth.

The operator $\mathcal{A}$ is the extended generator for the controlled Markov process [16,15]. If (A.1) holds then the stochastic process defined below is a martingale for initial conditions within $\mathcal{R}(\bar{q})$,

$$M_h(t) = h(X(t)) - h(X(0)) + \int_0^t \big(c(X(s)) - \phi\big)\, ds. \tag{A.2}$$

On defining the stopping times,

$$\begin{aligned} \tau_p &= \inf\{t \ge 0 : X(t) = (\bar{q}^p, 0)^T\}, \\ \tau_a &= \inf\{t \ge 0 : X(t) = (\bar{q}^a, 0)^T\}, \end{aligned} \tag{A.3}$$

we can construct a solution to (A.1) with quadratic growth through the conditional expectation,

$$h(x) = \mathsf{E}_x\Big[\int_0^{\tau_p} \big(c(X(t)) - \phi\big)\, dt\Big], \quad x \in \mathsf{X}. \tag{A.4}$$

For further details see [31,20] and [32, Chapter 17].

The differential generator is defined on C^2 functions via,

$$\mathcal{D}f := \langle \nabla f, Bu^a \rangle + \tfrac{1}{2}\sigma_D^2 \frac{\partial^2}{\partial q^2} f, \qquad \text{(A.5)}$$

with $u^a = (\zeta^{p+}, \zeta^{a+})^T$. If the C^2 function h satisfies $\mathcal{D}h = -c + \phi$ on $\mathcal{R}(\bar{q})$, and the boundary constraints,

$$\begin{aligned} \langle \nabla h(x), B1^1 \rangle = 0, \quad q = \bar{q}^p,\ g^a = 0, \\ \langle \nabla h(x), B1^2 \rangle = 0, \quad q = \bar{q}^a,\ g^a \geq 0, \end{aligned} \qquad \text{(A.6)}$$

it then follows from Itô's formula that the martingale property holds, so that (A.1) is satisfied. The C^2 condition can be relaxed by interpreting the equation $\mathcal{D}h = -c + \phi$ in the *viscocity* sense [17].

Proposition A.2 *Under any affine policy the function h given in (A.4) is well-defined and satisfies for some $b_0 < \infty$ and all x, $-b_0 \leq h(x) \leq b_0(\|q - g^a\|^2 + 1)$.*

The following bound is used twice in the proof of Proposition A.2:

Lemma A.3 *Suppose that $S \subset \mathsf{X}$ is a closed set satisfying, for some $T_0 < \infty$, $\varepsilon_0 > 0$,*

$$\mathsf{P}_x\{\tau_p \leq T_0\} \geq \varepsilon_0, \quad x \in S.$$

We then have for all $x \in \mathsf{X}$,

$$Z_x^p(S) := \mathsf{E}_x\Big[\int_0^{\tau_p} \mathbb{I}_S(X(t))\, dt\Big] \leq \varepsilon_0^{-1} T_0.$$

Proof: This is a generalization of [32, Lemma 11.3.10]. We have under the assumed bound,

$$\begin{aligned} Z_x^p(S) &\leq \varepsilon_0^{-1} \mathsf{E}\big[\textstyle\int_0^{\tau_p} \mathsf{P}_{X(t)}\{\tau_p \leq T_0\}\, dt\big] \\ &= \varepsilon_0^{-1} \textstyle\int_0^{\infty} \mathsf{E}\big[\mathbb{I}\{\tau_p \geq t\} \mathsf{P}_{X(t)}\{\tau_p \leq T_0\}\big]\, dt. \end{aligned} \qquad \text{(A.7)}$$

By the strong Markov property the expectation on the right hand side can be expressed,

$$\mathsf{E}\big[\mathbb{I}\{\tau_p \geq t\} \mathsf{P}_{X(t)}\{\tau_p \leq T_0\}\big] = \mathsf{E}\big[\mathbb{I}\{t \leq \tau_p \leq t + T_0\}\big].$$

Substituting this into (A.7) gives the desired bound. □

Proof of Proposition A.2: Consider the quadratic function $V_2(x) := \frac{1}{2}(q - g^a - \bar{q}^p)^2$, $x = (q, g^a)^T \in \mathsf{X}$. Applying the differential generator (A.5) to V_2 we obtain,

$$\mathcal{D}V_2(x) = \zeta^{p+}(q - g^a - \bar{q}^p) + \sigma_D^2, \quad x \in \mathcal{R}(\bar{q}).$$

Consideration of the gradient $\nabla V_2(x) = (q - g^a - \bar{q}^p)(1, -1)^T$, we see that the boundary conditions (A.6) hold, so that we deduce that V_2 is in the domain of the extended generator with $\mathcal{A}V_2(x) = \zeta^{p+}(q - g^a - \bar{q}^p) + \sigma_D^2$. From this identity it then follows that there exists a bounded set $S \subset \mathcal{R}(\bar{q})$, $\varepsilon_2 > 0$, and $b_2 < \infty$ such that (25) holds. This implies that the process is *c-regular* - see [31, Proposition 4.2]. Consequently, the upper bound on the solution to Poisson's equation follows directly from the main result of [20]. However, the method of proof in [20] does not consider this specific representation for the solution to Poisson's equation.

From the bound (25) we obtain the bound,

$$\mathsf{E}\Big[V_2(X(\tau_p)) + \varepsilon_2 \int_0^{\tau_p} c(X(s))\, ds\Big] \leq V(x) + b_2 Z_x^p(S) \qquad \text{(A.8)}$$

which when combined with Lemma A.3 implies the desired upper bound on h.

The lower bound on h follows from Lemma A.3 and the fact that c is *coercive*: the sublevel set $S_r = \{x : c(x) \leq r\}$ is a compact subset of $\mathcal{R}(\bar{q})$ for any r. Consequently, we have by the strong Markov property,

$$\inf_{x \in \mathsf{X}} h(x) = \inf_{x \in S_\phi} \mathsf{E}_x\Big[\int_0^{\tau_p} -\phi\, \mathbb{I}_{S_\phi}(X(s))\, ds\Big] > -\infty.$$

□

A.2 Invariance Equations

In this section we derive a number of steady-state formulae for the diffusion model under an affine policy with parameter $\bar{q} = (\bar{q}_p, \bar{q}_a)^T$.

Lemma A.4 $\mathsf{E}_\pi[G^a(t)] = \frac{\zeta^{a+}}{\zeta^{p+}} \mathsf{E}_\pi\big[|Q(t) - \bar{q}_a| \mathbb{I}(Q(t) < \bar{q}_a)\big]$

Proof: Define the two functions on X,

$$h_1(x) = g^a(q - \bar{q}_a), \quad h_2(x) = \tfrac{1}{2}(g^a)^2.$$

We then have by Itô's rule and the constraints on the process $dh_2(X) = G^a dG^a$, and

$$\begin{aligned} dh_1(X) &= (Q - \bar{q}_a) dG^a + G^a dQ \\ &= (Q - \bar{q}_a)\zeta^{a+} \mathbb{I}_{Q < \bar{x}_2}\, dt + G^a(dG^p + dG^a + dN). \end{aligned}$$

We can conclude that $T^{-1}\int_0^T G^a(t)dG^a(t)\,dt \to 0$ a.s., as $T\to\infty$, and

$$\frac{1}{T}\int_0^T (Q(t)-\bar q_a)\zeta^{a+}\mathbb{I}(Q(t)<\bar x_2)\,dt + \frac{1}{T}\int_0^T G^a(t)[dG^p(t)+dG^a(t)] \to 0.$$

Moreover, we have $G^a(t)dG^p(t) = G^a(t)\zeta^{p+}dt$ in view of the constraint that $G^a(t)=0$ if $Q(t)>\bar q_a$. Consequently, the limit theorems above combined with the ergodic theorem for $(\boldsymbol{Q},\boldsymbol{G}^a)$ give

$$\mathsf{E}\big[(Q(t)-\bar q_a)\zeta^{a+}\mathbb{I}(Q(t)<\bar x_2)+\zeta^{p+}G^a(t)\big]=0,$$

and this completes the proof. □

Lemma A.5 *The steady-state distribution of* $\boldsymbol{Q}$ *is conditionally exponential,*

$$\mathsf{P}_\pi\big\{Q(t)\le \bar q^a - q \mid Q(t)<\bar q^a\big\} = e^{-\gamma_0(\bar q^a-q)},\quad q\le \bar q^a.$$

Proof: Consider any fixed $\bar q\le\bar q_a$. To prove the lemma it is sufficient to establish the following identity for the conditional mean,

$$\mathsf{E}_\pi\big[|Q(t)-\bar q| \mid Q(t)<\bar q\big] = \gamma_0^{-1}. \tag{A.9}$$

Let $h(x)=\frac12(q-\bar q)^2\mathbb{I}(q<\bar q)$. This function is C^1, and we have by Itô's rule,

$$dh(X) = \zeta^+(Q-\bar q)\mathbb{I}(Q<\bar q)\,dt + (Q-\bar q)\mathbb{I}(Q<\bar q)\,dN + \tfrac12\sigma^2\mathbb{I}(Q<\bar q)\,dt$$

Applying the ergodic theorem as before then gives (A.9). □

Lemma A.6 *As* $t\to\infty$*, with probability one,*

$$t^{-1}I^p(t)\to\zeta^{p+},\quad t^{-1}I^a(t)\to\zeta^{a+}\mathsf{P}_\pi(Q(t)<\bar q_a).$$

Proof: We have for $T\ge0$,

$$G^a(T) = G^a(0) - I^a(T) + \int_0^T \zeta^{a+}\mathbb{I}(Q(t)<\bar q_a)\,dt\,.$$

Dividing both sides by T and letting $T\to\infty$ proves (ii). The proof of (i) identical, based on

$$Q(T) = Q(0)+\zeta^{p+}T - I^p(T) + G^a(T) + N(T).$$

□

Note that the limiting value of $T^{-1}I^p(T)$ does not depend upon the value of ζ^{a+}. This surprising result is consistent with the following formula for the density on $(\bar q_a,\bar q_p)$.

Lemma A.7 *The complementary distribution function* $\mathsf{P}_\pi\{Q(t)\le q\}$ *is given by (24).*

Proof: For $q<\bar q_a$ we can apply Lemma A.5. To establish the result for $q\in[\bar q_a,\bar q_p]$ we consider the test function

$$h(x) = [-\gamma_1(q-\bar q)+1-e^{-\gamma_1(q-\bar q)}]\mathbb{I}(q>\bar q)\,,$$

where $\bar q_a<\bar q<\bar q_p$ is fixed. We have by Itô's rule,

$$\begin{aligned} dh(X) &= [-\gamma_1+\gamma_1 e^{-\gamma_1(Q-\bar q)}]\mathbb{I}(Q>\bar q)dQ \\ &\quad +\tfrac12\sigma^2[-\gamma_1^2 e^{-\gamma_1(Q-\bar q)}]\,dt \\ &= -\gamma_1[1-e^{-\gamma_1(Q-\bar q)}]\mathbb{I}(Q>\bar q)[\zeta^{p+}dt - dI + dN] \\ &\quad -\tfrac12\gamma_1^2\sigma^2 e^{-\gamma_1(Q-\bar q)}]\,dt \\ &= -\gamma_1\zeta^{p+}\mathbb{I}(Q>\bar q)dt + \gamma_1[1-e^{-\gamma_1(\bar q_p-\bar q)}]\,dI \\ &\quad -\gamma_1[1-e^{-\gamma_1(Q-\bar q)}]\mathbb{I}(Q>\bar q)\,dN \end{aligned}$$

where in the last identity we use the formula for γ_1. Applying the ergodic theorem as above and applying Lemma A.6 then gives,

$$\mathsf{E}_\pi\big[-\gamma_1\zeta^{p+}\mathbb{I}(Q>\bar q)+\gamma_1\zeta^{p+}[1-e^{-\gamma_1(\bar q_p-\bar q)}]\big]=0\,,$$

and on canceling $\gamma_1\zeta^{p+}$ we obtain the result on $[\bar q_a,\bar q_p]$. □

Lemma A.8 $\mathsf{E}_\pi[Q(t)] = \bar q_p - \gamma_1^{-1} + \dfrac{\zeta^{a+}}{\zeta^{p+}}\mathsf{E}_\pi[|Q(t)-\bar q_a|\mathbb{I}(Q(t)<\bar q_a)]$.

Proof: Letting $h(x)=\frac12(q-\bar q_a)^2$ gives,

$$\begin{aligned} dh(X) &= (Q-\bar q_a)dQ + \tfrac12\sigma^2 dt \\ &= (Q-\bar q_a)[\zeta^{a+}\mathbb{I}(Q<\bar q_a)dt + dG^p + dN] + \tfrac12\sigma^2 dt \\ &= (Q-\bar q_a)[\zeta^{a+}\mathbb{I}(Q<\bar q_a)+\zeta^{p+}]\,dt \\ &\quad -(\bar q_p-\bar q_a)\,dI + (Q-\bar q_a)dN + \tfrac12\sigma^2 dt \end{aligned}$$

An application of Lemma A.6 and familiar arguments then gives,

$$\mathsf{E}_\pi\big[(Q-\bar q_a)[\zeta^{a+}\mathbb{I}(Q<\bar q_a)+\zeta\big]-(\bar q_p-\bar q_a)\zeta^{p+}+\tfrac12\sigma^2\big]=0.$$

Rearranging terms proves the result. □

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