An Extreme-Point Global Optimization Technique for Convex Hull Pricing

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***Abstract*—Prices in electricity markets are given by the dual variables associated with the supply-demand constraint in the dispatch problem. However, in unit-commitment-based day-ahead markets, these variables are less easy to obtain. A common approach relies on resolving the dispatch problem with the commitment decisions fixed and utilizing the associated dual variables. Yet, this avenue leads to inadequate revenues to generators and necessitates an *uplift payment* to be made by the market operator. Recently, a *convex hull pricing* scheme has been proposed to reduce the impact of such payments and requires the global maximization of the associated Lagrangian dual problem, which is, in general, a piecewise-affine concave function. In this paper, we present an extreme-point-based finite-termination procedure for obtaining such a global maximizer. Unlike standard subgradient schemes where an arbitrary subgradient is used, we present a novel technique where the *steepest* ascent direction is constructed by solving a continuous quadratic program. The scheme initiates a move along this direction with an a priori constant steplength, with the intent of reaching the boundary of the face. A backtracking scheme allows for mitigating the impact of excessively large steps. Termination of the scheme occurs when the set of subgradients contains the zero vector. Preliminary numerical tests are seen to be promising and display the finite-termination property. Furthermore, the scheme is seen to significantly outperform standard subgradient methods.**

***Index Terms*—Convex hull price, electricity markets, uplift payments, nondifferentiable optimization, Lagrangian relaxation.**

I. Introduction

CURRENTLY, all day-ahead electricity markets in North America adopt a unit-commitment-based model which explicitly takes into account each generator's physical/operational constraints and allows offers to recognize start-up, no-load and marginal costs. The operation and dispatch mechanisms of these markets are similar to those of a tight power pool under regulation: the quantity sold by each generator is determined by solving a centralized unit commitment and economic dispatch problem, except that the costs in this formulation are overridden by offer prices.

On the other hand, the price determination in these markets remains an open question. The common practice is to derive prices, or "marginal-cost prices", from the solution of the corresponding economic dispatch problem in which commitment decisions are fixed. Note that these commitment decisions are derived from an a priori solution of a unit commitment problem. As a consequence, commitment-dependent start-up and no-load costs are ignored in these prices and the payments collected from the auction based on such prices may be insufficient to compensate the generators. To overcome this problem, *uplift payment* mechanisms have been introduced, through which additional side payments are made to the generators in recognition of the costs incurred due to commitment decisions. A possibly large amount of uplift payments is problematic for both the operator and the market participants, since they are neither transparent nor easy to justify. Furthermore, it is observed that marginal-cost prices are no longer monotonically increasing with demand. For instance, a high price does not necessarily indicate a relatively high demand level; instead, it may arise from the fact that generators with high offers are selected and happen to set the hourly prices. Accordingly, prices fail to assume their usual economic roles as reporters of the alignment between supply and demand, or incentives for adjusting supply and consumption levels [1]. Additionally, the strong dependence of marginal-cost prices on the commitment solution raise concerns regarding the equity, efficiency and economic rationale of electricity markets [2], [3]. A further concern is that size of the unit commitment problems often prevents its exact solution. Often, the dual variables associated with the corresponding dispatch problems, parameterized by these commitment decisions, may vary dramatically.

"Convex hull pricing" has been suggested as an alternative pricing scheme to overcome or mitigate these undesired properties [4], [5]. Instead of fixing the commitment decisions, the convex hull price scheme is derived in a fashion in which the commitment-dependent start-up and no-load costs are explicitly considered. In fact, convex hull prices are non-decreasing with respect to the demand, and lead to the minimal amount of total "opportunity-cost" uplift payments [5]. Recently, it has been shown that the solution to the convex hull price problem can be obtained by solving the corresponding Lagrangian dual of the the unit commitment problem [5]. The dual problem has been extensively studied in Lagrangian-relaxation framework in last several decades [6]. One scheme for solving this dual is the subgradient-based method. In such techniques, subgradients in the dual space are obtained by evaluating constraint violations in the primal space where the associated primal variables are obtained by minimizing the Lagrangian in the primal space. Under a diminishing

Research supported in part by the Grainger Endowments to the University of Illinois. Gui Wang and Sean Meyn are with the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, IL 61801 USA. Uday V. Shanbhag is with the Department of Industrial and Enterprise Systems Engineering at the same university. Tongxin Zheng and Eugene Litvinov are with Business Architecture and Technology, ISO New England, Holyoke, MA 01040 USA. Disclaimer: The views expressed in this paper are solely those of the authors and not necessarily those of ISO New England.

steplength rule, the convergence of subgradient methods is guaranteed. In practice, such techniques often tend to have difficulty when the dual function is ill-conditioned and local convergence behavior can deteriorate significantly. Extensions of such schemes lie in the use of bundle methods that incorporate memory in constructing convex combinations over a historically aggregated set of subgradients. These issues are exacerbated in power system applications where degeneracy is a commonly observed challenge [7]. Various enhancements has been incorporated into the subgradient-based framework to improve the efficiency, such as the use of cutting-plane methods [8], interior-point methods [9], bundle methods [10], and surrogate gradient methods [11].

To the best of our knowledge, References [12], [13] are the only papers focusing specifically on the computational tools for convex hull pricing. In their pioneering work, subgradient and simplex techniques are adopted in central cutting-plane methods to generate "interior" points with less computational burden and redundant constraints are pruned to further increase efficiency. By shrinking the feasible region, such methods are expected to avoid convergence to non-optimal solutions. Yet, convergence behavior of this scheme appears to still be less favorable, as parallel cutting planes are generated through the iterations (see Fig 7 [13]).

The remainder of this paper contains five additional sections and is organized as follows. In Section II, we present a mathematical model for convex hull pricing. Section III is devoted to analyzing the structure of the convex hull pricing problem and the structural characteristics are exploited in Section IV to develop an effective and efficient algorithm to compute convex hull prices. Numerical results of the proposed method are presented in Section V and the paper concludes with some remarks and final thoughts in Section VI.

II. Convex Hull Pricing Model

In this section, we formulate a mathematical model of day-ahead markets currently prevalent in North America, based on which convex hull prices are defined and studied. Consider an H period Day-Ahead Market (DAM) with S generators. Let $f_s(\underline{\boldsymbol{u}}_{\boldsymbol{s}}, \underline{\boldsymbol{p}}_{\boldsymbol{s}})$ be generator s's offer function, where $\underline{\boldsymbol{u}}_{\boldsymbol{s}} \in \{0,1\}^H$ denotes the on/off or commitment status of generator s over the H periods. Furthermore, $\underline{\boldsymbol{p}}_{\boldsymbol{s}} \in (\{0\} \cup [p_s^{min}, p_s^{max}])^H$ denotes generator s's energy dispatch levels, which could be zero (if off) or between p_s^{min} and p_s^{max}, the maximal and minimal power output levels of generator s. Currently, all DAMs in the US adopt an offer format where $f_s(\underline{\boldsymbol{u}}_{\boldsymbol{s}}, \underline{\boldsymbol{p}}_{\boldsymbol{s}})$ is piecewise linear with respect to $\underline{\boldsymbol{p}}_{\boldsymbol{s}}$ and we use the same offer format in our model.

Let X_s be the polyhedral operational region defined by resource-based physical and/or operational constraints imposed on generator s. Let $\underline{\boldsymbol{d}} \in R^H$ denote the demand vector over the H periods. Then the Unit Commitment Problem (UCP) requires a set of commitment and dispatch decisions to satisfy the demand in the least "bid cost" manner, while being feasible with respect to physical and operational constraints.

Definition 1 (UCP)**.** *The UCP is defined as*

$$\begin{array}{lll} \min\limits_{\underline{\boldsymbol{u}}_{s}, \underline{\boldsymbol{p}}_{s}, \forall s} & \sum_{s=1}^{S} f_s(\underline{\boldsymbol{u}}_{\boldsymbol{s}}, \underline{\boldsymbol{p}}_{\boldsymbol{s}}) & \\ st: & \sum_{s=1}^{S} \underline{\boldsymbol{p}}_{\boldsymbol{s}} = \underline{\boldsymbol{d}}, & \\ & (\underline{\boldsymbol{u}}_{\boldsymbol{s}}, \underline{\boldsymbol{p}}_{\boldsymbol{s}}) \in \mathsf{X}_s, & \forall s. \end{array} \tag{1}$$

We define the value function as the value of the UCP, parameterized by the demand, as follows:

Definition 2 (Value Function)**.** *The value function of the UCP is defined as*

$$v(\underline{\boldsymbol{d}}) \triangleq \min_{(\underline{\boldsymbol{u}}_{s}, \underline{\boldsymbol{p}}_{s}) \in \mathsf{X}_s, \forall s} \left\{ \sum_{s=1}^{S} f_s(\underline{\boldsymbol{u}}_{\boldsymbol{s}}, \underline{\boldsymbol{p}}_{\boldsymbol{s}}) | \sum_{s=1}^{S} \underline{\boldsymbol{p}}_{\boldsymbol{s}} = \underline{\boldsymbol{d}} \right\}. \tag{2}$$

A salient characteristic of the value function is that, on the set of $\underline{\boldsymbol{d}}$ for which (UCP) is feasible, the value function is lower semicontinuous and differentiable almost everywhere. Indeed, the widely adopted marginal cost pricing model uses gradient or subgradient information associated with the value function as a candidate price.

On the other hand, the convex hull pricing scheme derives prices from the convex hull of the value function rather than the value function itself. The convex hull of a nonconvex function is the largest convex function that does not exceed the given function in value at any given point in the domain [14] and is formally defined next.

Definition 3 (Convex Hull of the Value Function)**.** *The convex hull of $v(\underline{\boldsymbol{d}})$ is defined as*

$$v^h(\underline{\boldsymbol{d}}) \triangleq \inf\{\mu | (\underline{\boldsymbol{d}}, \mu) \in conv(epi(v(\underline{\boldsymbol{d}})))\}, \tag{3}$$

where $epi(f)$ is the epigraph of a function f and $conv(K)$ denotes the convex hull of K.

The convex hull price is defined next.

Definition 4 (Convex Hull Price)**.** *The convex hull price, denoted by $\underline{\boldsymbol{\rho}}^{\boldsymbol{h}}(\underline{\boldsymbol{d}})$, is defined as the subgradient of the convex hull of the value function or*

$$\underline{\boldsymbol{\rho}}^{\boldsymbol{h}}(\underline{\boldsymbol{d}}) \in \partial v^h(\underline{\boldsymbol{d}}). \tag{4}$$

To simplify notation, we suppress the dependence of the convex hull price on $\underline{\boldsymbol{d}}$ in the rest of this paper.

III. Analysis of Convex Hull Pricing Problem

In this section, we investigate the structural characteristics of the convex hull price problem for DAMs.

A. Convex Hull Price and Lagrangian Dual Problem

Obtaining subgradients of $v^h(\underline{\boldsymbol{d}})$ is a challenging proposition, since it necessitates computing the convex hull of a function. To exacerbate matters, every point in the hull requires the solution of a unit commitment problem, or effectively a mixed-integer linear program. An alternate approach lies in solving the Lagrangian Dual Problem instead.

Definition 5 (Lagrangian Dual Problem of the UCP)**.** *Suppose the Lagrangian dual function, $L(\underline{\boldsymbol{\rho}})$ is defined as*

$$L(\underline{\boldsymbol{\rho}}) \triangleq \min_{(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) \in \mathsf{X}_s, \forall s} \left\{ \sum_{s=1}^{S} f_s(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) + \underline{\boldsymbol{\rho}}^T \left(\underline{\boldsymbol{d}} - \sum_{s=1}^{S} \underline{\boldsymbol{p}}_s \right) \right\}.$$

Then the Lagrangian dual problem is given by

$$\max_{\underline{\boldsymbol{\rho}}} L(\underline{\boldsymbol{\rho}}). \tag{5}$$

The relationship between the Lagrangian dual problem and the original convex hull pricing problem is made precise by the next proposition [5], which is provided without a proof.

Proposition 6. *Let $\underline{\boldsymbol{\rho}}^*$ be an optimal solution to* (5)*. Then*

$$\underline{\boldsymbol{\rho}}^* \in \partial v^h(\underline{\boldsymbol{d}}). \tag{6}$$

B. Characteristics of the Lagrangian Dual Problem

It is obvious that the Lagrangian dual function, $L(\underline{\boldsymbol{\rho}})$, is separable with respect to generators, and thus, the computation of $L(\underline{\boldsymbol{\rho}})$ can be reduced to a collection of S sub-problems. Furthermore, the Lagrangian dual function is concave and allows for the use of cutting plane methods.

This paper stresses the problem characteristics resulting from the piecewise linear format of the offers. Indeed, the the piecewise linear format is part of the market design to make the market clearing computationally possible. For the error analysis of such approximation, we refer the readers to [15]. To facilitate our discussion, we define two sets as follow.

Definition 7 (Generators's Auction-Surplus-Maximization Quantity Set)**.** *Given a price vector $\underline{\boldsymbol{\rho}}$, the auction-surplus-maximization quantity set associated with generator s is defined as*

$$B_s(\underline{\boldsymbol{\rho}}) \triangleq \{\underline{\boldsymbol{p}}_s | (\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) \in \arg\max_{(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) \in \mathsf{X}_s} [\underline{\boldsymbol{\rho}}^T \underline{\boldsymbol{p}}_s - f_s(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s)]\}. \tag{7}$$

Note that the optimization problem specified in (7) involves H binary and H continuous decision variables. Thus, we may use `CPLEX` [16], a commercial solver for large-scale mixed-integer linear and quadratic programs, capable of generating and storing multiple solutions [16], to obtain extreme points of $B_s(\underline{\boldsymbol{\rho}})$.

Definition 8 (Aggregated Auction-Surplus-Maximization Quantity Set)**.** *Given a price vector $\underline{\boldsymbol{\rho}}$, the aggregated auction-surplus-maximization quantity set is defined as*

$$B(\underline{\boldsymbol{\rho}}) \triangleq \left\{ \sum_{s=1}^{S} \underline{\boldsymbol{p}}_s | \underline{\boldsymbol{p}}_s \in B_s(\underline{\boldsymbol{\rho}}), s = 1, \ldots, S \right\}. \tag{8}$$

Note that $B_s(\underline{\boldsymbol{\rho}})$ and $B(\underline{\boldsymbol{\rho}})$ may be infinite, as a consequence of the degeneracies in the auction-surplus maximization problem. For example, if price coincides with marginal cost for a certain segment of a generator's piecewise linear offer, then the generator is indifferent towards operating at any point within the segment. As a result, the generator's auction surplus may be maximized over an interval, rather than a point. Fortunately, given the piecewise linear format, no matter what prices may be, the convex hull of the auction-surplus-maximization quantity sets has a finite number of extreme points. As a consequence, the maximization problem in (7) becomes a finite-dimensional linear program, once the commitment decisions, denoted by $\underline{\boldsymbol{u}}_s$, are given. The feasible region of such an LP is determined by $\underline{\boldsymbol{u}}_s$ and X_s. Let $\Theta(\underline{\boldsymbol{u}}_s)$ denote the set of all vertices of the feasible region. Then, a solution to the linear program is attained at either a vertex, which is in $\Theta(\underline{\boldsymbol{u}}_s)$, or a convex combination of vertices, which is in $conv(\Theta(\underline{\boldsymbol{u}}_s))$. Define Φ_s and Φ as

$$\Phi_s \triangleq \cup_{\forall \underline{\boldsymbol{u}}_s} \Theta(\underline{\boldsymbol{u}}_s) \text{ and } \Phi \triangleq \left\{ \sum_{s=1}^{S} \underline{\boldsymbol{p}}_s | \underline{\boldsymbol{p}}_s \in \Phi_s, \forall s \right\}, \tag{9}$$

respectively. Next, we relate the convex hulls of B_s and B to $B_s \cap \Phi_s$ and $B \cap \Phi$.

Lemma 9. *Suppose Φ_s and Φ are defined by* (9)*. Then, Φ_s and Φ are finite. Furthermore, $conv(B_s(\underline{\boldsymbol{\rho}})) = conv(B_s(\underline{\boldsymbol{\rho}}) \cap \Phi_s)$ and $conv(B(\underline{\boldsymbol{\rho}})) = conv(B(\underline{\boldsymbol{\rho}}) \cap \Phi)$.*

Proof: The finiteness of Φ_s follows from the finite number of vertices in a finite-dimensional LP. Since the number of commitment decisions is finite, Φ_s is also finite since the union is over a finite set.

Since $B_s \cap \Phi \subseteq B_s$, it follows that $conv(B_s(\underline{\boldsymbol{\rho}})) \supseteq conv(B_s(\underline{\boldsymbol{\rho}}) \cap \Phi_s)$. It suffices to show that $conv(B_s(\underline{\boldsymbol{\rho}})) \subseteq conv(B_s(\underline{\boldsymbol{\rho}}) \cap \Phi_s)$. Remember $B_s(\underline{\boldsymbol{\rho}}) \cap \Phi_s$ gives all extreme-point solutions. By the fundamental theorem of linear programming, $B_s(\underline{\boldsymbol{\rho}}) \subseteq conv(B_s(\underline{\boldsymbol{\rho}}) \cap \Phi_s)$, consequently, $conv(B_s(\underline{\boldsymbol{\rho}})) \subseteq conv(B_s(\underline{\boldsymbol{\rho}}) \cap \Phi_s)$. Likewise, we can prove the same conclusions hold for Φ, which is obtained by aggregating elements in each generator's $B_s(\underline{\boldsymbol{\rho}})$. ■

We refer the readers to [14] for a general proof of the concavity of the dual. The concavity and piecewise linearity of $L(\underline{\boldsymbol{\rho}}))$ for the present problem is shown next. Note that it is likely that such a result may have been proved elsewhere, given its simplicity, yet we have no precise reference.

Lemma 10. *Consider $L(\underline{\boldsymbol{\rho}})$ as defined in Def. 5. Then $L(\underline{\boldsymbol{\rho}})$ is a concave and piecewise linear function of $\underline{\boldsymbol{\rho}}$.*

Proof: By definition, $L(\underline{\boldsymbol{\rho}})$ is given by

$$\begin{aligned}
&= \min_{(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) \in \mathsf{X}_s, \forall s} \left\{ \textstyle\sum_{s=1}^{S} f_s(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) + \underline{\boldsymbol{\rho}}^T (\underline{\boldsymbol{d}} - \sum_{s=1}^{S} \underline{\boldsymbol{p}}_s) \right\} \\
&= \underline{\boldsymbol{\rho}}^T \underline{\boldsymbol{d}} + \textstyle\sum_{s=1}^{S} \min_{(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) \in \mathsf{X}_s} \left\{ f_s(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) - \underline{\boldsymbol{\rho}}^T \underline{\boldsymbol{p}}_s \right\} \\
&= \underline{\boldsymbol{\rho}}^T \underline{\boldsymbol{d}} + \textstyle\sum_{s=1}^{S} \min_{(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) \in \mathsf{X}_s, \underline{\boldsymbol{p}}_s \in \Phi_s} \left\{ f_s(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) - \underline{\boldsymbol{\rho}}^T \underline{\boldsymbol{p}}_s \right\}.
\end{aligned}$$

where the last equality holds since the optimum occurs at least one of the vertices. Note for any given elements in Φ_s, $f_s(\underline{\boldsymbol{u}}_s, \underline{\boldsymbol{p}}_s) - \underline{\boldsymbol{\rho}}^T \underline{\boldsymbol{p}}_s$ becomes an affine function with respect to $\underline{\boldsymbol{\rho}}$. Therefore $L(\underline{\boldsymbol{\rho}})$ is a point-wise minimization of finite number of linear functions, which is concave and piecewise linear. ■

The maximization of a concave and piecewise linear function may be formulated as an LP, whose optimum, if it exists, is always attained at extreme points; however, the constraint set requires enumerating all possible extreme points and such an avenue is generally inadvisable.

Proposition 11 (Necessary and Sufficient Optimality Conditions for Convex Hull Price)**.** *Given a demand vector $\underline{\boldsymbol{d}}$, $\underline{\boldsymbol{\rho}}$ is a convex hull price for $\underline{\boldsymbol{d}}$ if and only if*

$$\underline{\boldsymbol{d}} \in conv(B(\underline{\boldsymbol{\rho}})). \tag{10}$$

Proof: From Lemma 10, for any element in Φ_s, $f_s(\underline{\boldsymbol{u}}_{\boldsymbol{s}}, \underline{\boldsymbol{p}}_{\boldsymbol{s}}) - \underline{\boldsymbol{\rho}}^T \underline{\boldsymbol{p}}_{\boldsymbol{s}}$ is an affine function of $\underline{\boldsymbol{\rho}}$ with gradient $-\underline{\boldsymbol{p}}_{\boldsymbol{s}}$. Also, because set Φ_s is finite, by Corollary 2.6 in [17], we obtain

$$\begin{aligned} \partial L(\underline{\boldsymbol{\rho}}) &= \underline{\boldsymbol{d}} - \textstyle\sum_{s=1}^{S} conv\{\underline{\boldsymbol{p}}_{\boldsymbol{s}} | \underline{\boldsymbol{p}}_{\boldsymbol{s}} \in B_s(\underline{\boldsymbol{\rho}})\} \\ &= \{\underline{\boldsymbol{d}} - \hat{\underline{\boldsymbol{d}}} | \hat{\underline{\boldsymbol{d}}} \in conv(B(\underline{\boldsymbol{\rho}}))\}. \end{aligned}$$

The necessary and sufficient conditions for $\underline{\boldsymbol{\rho}}$ to be optimal are $\underline{\boldsymbol{0}} \in \partial L(\underline{\boldsymbol{\rho}})$ [18], which is equivalent to $\underline{\boldsymbol{d}} \in conv(B(\underline{\boldsymbol{\rho}}))$. ■

IV. Algorithm

Based on the analysis in Section III, it emerges that a convex hull price, denoted by $\underline{\boldsymbol{\rho}}$, is obtained when the load vector lies in the convex hull of $B(\underline{\boldsymbol{\rho}})$. Intuitively, if this were not the case, then this suggests a method in which the convex hull of the aggregate auction-surplus-maximization quantity set is iteratively moved towards the demand vector by updating the price vector appropriately. In this section, we begin by providing an outline of the algorithm and how search directions for $\underline{\boldsymbol{\rho}}$ may be efficiently computed. Then, we provide a brief overview of steplength choice and backtracking schemes. The section concludes with an illustrative example.

A. Algorithm Structure

In iterative algorithms, one often uses a merit function to obtain a measure of progress. In this instance, an appropriate choice would be the Euclidean distance between the demand and the convex hull of the aggregated auction-surplus-maximization quantity set. Notably, this merit function is nonnegative and is zero at optimality.

Definition 12 (Distance between Demand and Convex Hull of Aggregated Auction-Surplus-Maximization Quantity Set)**.** *The distance between $\underline{\boldsymbol{d}}$ and $conv(B(\underline{\boldsymbol{\rho}}))$, denoted by $\gamma(\underline{\boldsymbol{\rho}})$, is defined as*

$$\gamma(\underline{\boldsymbol{\rho}}) \triangleq \|\underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}) - \underline{\boldsymbol{d}}\|, \tag{11}$$

where $\underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}})$ is the projection of $\underline{\boldsymbol{d}}$ onto $conv(B(\underline{\boldsymbol{\rho}}))$, and defined as

$$\underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}) \triangleq \underset{\hat{\underline{\boldsymbol{d}}}(\underline{\boldsymbol{\rho}}) \in conv(B(\underline{\boldsymbol{\rho}}))}{\arg\min} \|\underline{\boldsymbol{d}} - \hat{\underline{\boldsymbol{d}}}\|. \tag{12}$$

.

Since $\underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}})$ is a projection of a point onto a convex set, it is always uniquely defined. Furthermore, the hyperlane $\{\underline{\boldsymbol{x}} | (\underline{\boldsymbol{d}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}))^T (\underline{\boldsymbol{x}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}})) = 0\}$ separates the quantity space into two half-spaces: $\{\underline{\boldsymbol{x}} | (\underline{\boldsymbol{d}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}))^T (\underline{\boldsymbol{x}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}})) \geq 0\}$ and $\{\underline{\boldsymbol{x}} | (\underline{\boldsymbol{d}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}))^T (\underline{\boldsymbol{x}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}})) \leq 0\}$. Furthermore, since the hyperlane also supports $conv(B(\underline{\boldsymbol{\rho}}))$, we have that $conv(B(\underline{\boldsymbol{\rho}}))$ belongs to the latter half-space implying that $B(\underline{\boldsymbol{\rho}})$ also belongs to the latter. We propose to update the prices along the direction of vector $\underline{\boldsymbol{d}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}})$, which coincides with the steepest ascent direction.

Proposition 13. *The vector $\underline{\boldsymbol{d}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}})$ is the steepest ascent direction of $L(\underline{\boldsymbol{\rho}})$.*

Proof: See Theorem 1.11 in [18]. ■

Note although the steepest ascent direction is selected as the search direction in the scheme, our main purpose is not to make the largest progress in terms of $L(\underline{\boldsymbol{\rho}})$. Due to the non-differentiable nature of the $L(\underline{\boldsymbol{\rho}})$, myopic greedy algorithm will lead to zigzag and introduce computational difficulties. Instead, we choose such a direction because decrease in γ is always possible along the steepest ascent direction until it reaches zero, which is sufficient and necessary for global optimality.

We propose to compute the steepest ascent direction by solving the projection problem in (12). This is generally a challenging task since it requires the computation of $conv(B(\underline{\boldsymbol{\rho}}))$. To facilitate this computation, we employ the extreme points of $conv(B(\underline{\boldsymbol{\rho}}))$, which in turn require the solution of a large-scale MIP problem. In order to further reduce the computational burden, we obtain the extreme points of $conv(B(\underline{\boldsymbol{\rho}}))$ as an aggregation of the extreme points of $conv(B_s(\underline{\boldsymbol{\rho}}))$.

Define $\{p_{sk}\} \triangleq B_s(\underline{\boldsymbol{\rho}}) \cap \Phi_s$ with cardinality K_s, then by Lemma 9, it contains all extreme points of $conv(B_s(\underline{\boldsymbol{\rho}}))$. By doing this, we solve S smaller MIP sub-problems, one for each generator, in each iteration instead of a large MIP problem over S generators. Note that even with an extreme point representation of $conv(B(\underline{\boldsymbol{\rho}}))$, the number of extreme points may grow to an exponential level as the the number of generators increases. The resulting projection problem can be cast as the following convex quadratic program.

$$\begin{array}{lll} & \underset{\lambda_{sk}, \forall s,k}{\min} & \|\underline{\boldsymbol{d}} \quad - \sum_{s=1}^{S} \sum_{k=1}^{K_s} \lambda_{sk} p_{sk}\|^2 \\ st: & \sum_{k=1}^{K_s} \lambda_{sk} & = 1, \forall s \\ & \lambda_{sk} & \geq 0, \forall s, k. \end{array} \tag{13}$$

.

The solution to the above problem, in fact, gives the projection we need.

Proposition 14. *Let $\{\lambda_{sk}\}$ solves (13), then $\sum_{s=1}^{S} \sum_{k=1}^{K_s} \lambda_{sk} p_{sk}$ solves (12).*

Proof: The proposition holds because the convex hull operation and the sum are commutable. Therefore,

$$\begin{aligned} conv(B(\underline{\boldsymbol{\rho}})) &= conv(\left\{ \sum_{s=1}^{S} \underline{\boldsymbol{p}}_{\boldsymbol{s}} | \underline{\boldsymbol{p}}_{\boldsymbol{s}} \in B_s(\underline{\boldsymbol{\rho}}), \forall s \right\}) \\ &= \left\{ \sum_{s=1}^{S} \underline{\boldsymbol{p}}_{\boldsymbol{s}} | \underline{\boldsymbol{p}}_{\boldsymbol{s}} \in conv(B_s(\underline{\boldsymbol{\rho}})), \forall s \right\} \\ &= \left\{ \sum_{s=1}^{S} \underline{\boldsymbol{p}}_{\boldsymbol{s}} | \underline{\boldsymbol{p}}_{\boldsymbol{s}} \in conv(B_s(\underline{\boldsymbol{\rho}}) \cap \Phi_s), \forall s \right\} \\ &= \left\{ \sum_{s=1}^{S} \sum_{k=1}^{K_s} \lambda_{sk} p_{sk} | \sum_{k=1}^{K_s} \lambda_{sk} = 1, \forall s; \lambda_{sk} \geq 0, \forall s, k \right\}. \end{aligned}$$

■

The number of decision variables in (13), $\sum_{s=1}^{S} K_s$, increases at a linear rate with respect to the number of generators. Given the offers from electricity markets under normal conditions, $K_s = 1$ for most generators: due to the large distinctions of different generating technologies in terms of economic merits and operational flexibility, most generators are either operated at the maximum, or priced themselves out of the markets. Accordingly, the scale of (13) remains small and can be easily solved.

We are now ready to state our algorithm.

Algorithm 15 (Extreme-Point-Based Global Optimization Technique)**.**

Initialization: price $\underline{\boldsymbol{\rho}}^0$, max steplength $c > 0$; iteration index $\nu = 1$; merit measure $\gamma^0 = \infty$; search direction $\underline{\boldsymbol{\Delta}}^0 = \underline{\boldsymbol{0}}$.
while $\gamma^\nu \neq 0$ **do**
    Obtain $B_s(\underline{\boldsymbol{\rho}})$ by solving (7) for each s;
    Compute $\underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^\nu)$ by solving (13);
    Set $\gamma^\nu = \|\underline{\boldsymbol{d}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^\nu)\|$.
    if $\gamma^\nu < \gamma^{\nu-1}$, or $\underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^\nu) = \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^{\nu-1})$ **then**
        $\alpha^\nu = c$, $\underline{\boldsymbol{\Delta}}^\nu = \dfrac{\underline{\boldsymbol{d}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^\nu)}{\|\underline{\boldsymbol{d}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^\nu)\|}$;
        $\underline{\boldsymbol{\rho}}^{\nu+1} = \underline{\boldsymbol{\rho}}^\nu + \alpha^\nu \underline{\boldsymbol{\Delta}}^\nu$.
    else
        Compute α^ν via linesearch along $\underline{\boldsymbol{\Delta}}^{\nu-1}$ such that an acceptable descent is obtained in γ^ν (Algo. 16).
    end if
    $\nu = \nu + 1$;
end while

B. Steplength Selection

Standard subgradient schemes rely on diminishing steplength rules that require a steplength sequence satisfying $\sum_{\nu=1}^{\infty} \alpha^\nu = \infty$ and $\sum_{\nu=1}^{\infty} (\alpha^\nu)^2 < \infty$. Here, we choose an alternate approach that takes advantage of the fact that we obtain a descent direction with respect to γ^ν. Note that in general, such a descent requirement is not guaranteed when an arbitrary subgradient is selected. Given such a property, the scheme employs a linesearch along the obtained search direction to ensure a monotonic decrease in γ.

When using Newton-based methods, the default steplength is unity; this is chosen as the point from which the steplength may be reduced till sufficient descent is made with respect to a suitably defined merit function. In this setting, a user-specified upper bound, denoted by c, is employed. If such a steplength leads to an increase in γ, a backtracking-based linesearch is adopted to generate a strict decrease of γ.

For the proposed backtracking scheme method to make progress, we need to guarantee that (1) $\underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^\nu) = \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^{\nu-1})$ will not happen infinitely often (cycling); and, (2) when backtracking is adopted, α^ν leading a strict decrease in γ, always exists and can be obtained. The first condition holds when the primal UCP is feasible. Suppose $\underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^\nu) = \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^{\nu-1})$ forever, then the method will always update the price along the same direction with a constant and positive steplength, and this direction is always the steepest ascent direction at each iteration. This only happens when either $L(\underline{\boldsymbol{\rho}})$ goes to infinity, or the global optimum is already achieved, because of the fact that $L(\underline{\boldsymbol{\rho}})$ is concave and piecewise linear. However, feasibility of the UCP imposes upper bound on $L(\underline{\boldsymbol{\rho}})$ and if the global optimum is achieved, $\gamma^\nu = 0$, and the algorithm terminates. As mentioned earlier, the existence of a suitable steplength is rooted in the fact that the search direction is the steepest descent direction. Our exposition will be restricted towards discussing the mechanics of our scheme, rather than the theoretical underpinnings.

Algorithm 16 (Backtracking Linesearch Scheme)**.**

Initialization. Set $flag = \nu - 1$; get $\underline{\boldsymbol{\Delta}}^{flag}$ and $\underline{\boldsymbol{\rho}}^{flag}$ from the main loop of the proposed algorithm;
while $\gamma^\nu \geq \gamma^{flag}$ **do**
    Obtain α by solving

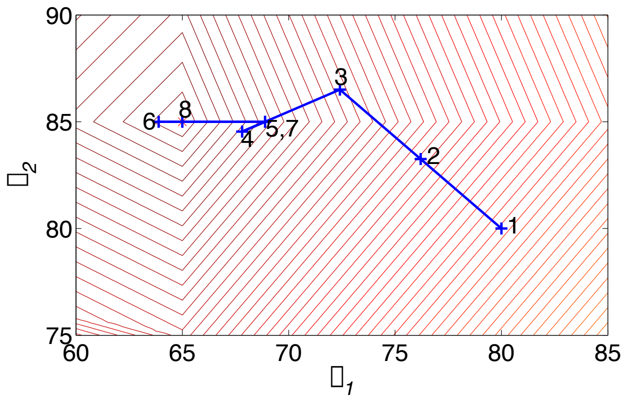
$$\begin{aligned}&(\underline{\boldsymbol{\rho}}^{flag} + \alpha(\underline{\boldsymbol{\Delta}}^{flag})^T \underline{\boldsymbol{d}}^\nu - v(\underline{\boldsymbol{d}}^\nu)\\ =&(\underline{\boldsymbol{\rho}}^{flag} + \alpha(\underline{\boldsymbol{\Delta}}^{flag})^T \underline{\boldsymbol{d}}^{flag} - v(\underline{\boldsymbol{d}}^{flag});\end{aligned} \tag{14}$$

    Set $\underline{\boldsymbol{\rho}}^{\nu+1} = \underline{\boldsymbol{\rho}}^{flag} + \alpha \underline{\boldsymbol{\Delta}}^{flag}$, $\nu = \nu + 1$;
    Obtain $B_s(\underline{\boldsymbol{\rho}})$ by solving (7) for each s;
    Compute $\underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^\nu)$ by solving (13);
    Set $\gamma^\nu = \|\underline{\boldsymbol{d}} - \underline{\boldsymbol{d}}^{\boldsymbol{p}}(\underline{\boldsymbol{\rho}}^\nu)\|$.
end while

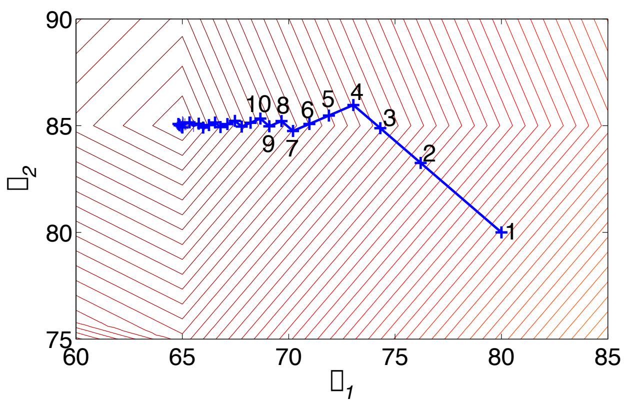
The linesearch is triggered if the new candidate iterate is in a new face and an increase in γ. In this case, the faces of $L(\underline{\boldsymbol{\rho}})$ that are crossed by this overshooting are investigated in the hope of reducing γ. The next example illustrates the scheme.

C. An Illustrative Example

To illustrate how the proposed procedure works, we plot the trajectory of iterates for a two-hour example in the space of $L(\underline{\boldsymbol{\rho}})$, as shown in Figure 1(a). Since $L(\underline{\boldsymbol{\rho}})$ is piecewise linear, the 4 sets of parallel lines actually represents 4 faces of $L(\underline{\boldsymbol{\rho}})$. As shown in the plot, starting from prices of 80 \$/MWh for both hours, the proposed method attain the global optimum after 8 iterations. Note that $L(\underline{\boldsymbol{\rho}})$ is differentiable at the first 4 iterates, and these points are generated along the gradient direction. The 4th iterate returns to the same face of $L(\underline{\boldsymbol{\rho}})$ as the starting point. Then backtracking is triggered, and with reduced steplength, the 5th iterate is reached and lies at the intersection of the two faces. In traditional sub-gradient methods without backtracking, the iterates will oscillate between these faces. From the 5th iterate, the scheme determines the steepest ascent direction that lies at the intersection of the faces. Then, the scheme overshoots the optimal solution. From the 6th iterate, the scheme returns to the 7th point which is identical to the 5th. However, since the merit function increases upon the return, backtracking is triggered and based on the information of the 6th and 7th query points, the global optimum is achieved in the next iteration. Note that when we employ a standard subgradient scheme as in Figure 1(b), we observe that the scheme takes a significantly longer time to converge. The efficiency improvement can be largely assigned



(a)



(b)

Fig. 1. (a) trajectory of query points of proposed method; (b) trajectory of query points of subgradient method

to the improvements in steplength and direction selection. Geometrically, the algorithm updates solutions with the *explicit* purpose of evaluating the extreme points of the dual function.

V. Numerical Testing

We present preliminary numerical results for the proposed method in this section. All the numerical tests were carried out on a PC with Intel Core i5 2.66 GHz processor with 6 GB RAM. We first use the proposed method to solve the three-generator testing problems given in [13]. These examples are carefully designed such that the duality gaps are always zero. In these problems, there are no inter-temporal constraint, such as the minimal up/down time constraints. We compare the performance of the proposed method to the results of subgradient-simplex based cutting plane method (SSCPM) and the analytic center cutting plane method (ACCPM) from [13], which, to the best of our knowledge, are the only methods explicitly designed to solve the convex hull pricing problem. We compare the number of iterations as a measure of efficiency, because the complexity of each iteration is almost the same for all subgradient-based methods. Therefore, we expect the CPU time to be proportional to the number of iterations. The initial guess of the convex hull prices is set to be 80 \$/MWh for all hours in these examples, and the maximal steplength is

	SSCPM	ACCPM	Proposed method
2-h problem	41	–	8
10-h problem	75	216	38
24-h problem	347	300	61

TABLE I
Number of iterations to converge to global optimum

5 for these examples. The optimal solutions are omitted here since the proposed method achieves the same solutions given in [13]. The comparative results given in Table I show that the proposed method converges much faster than both SSCPM and ACCPM.

We also compare the proposed method for the 24-hour problem with the general subgradient method and the steepest ascent method with $20/\nu$ diminishing steplength. Results of the first 300 iterations are shown since these general methods do not converge after an acceptable number of iterations or converge to non-optimal points. In Figure 2, all methods

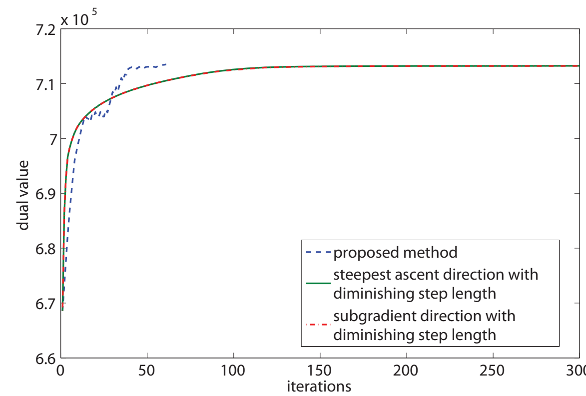


Fig. 2. Value of the dual function through iterations

nearly obtain the global optimal dual value. If obtaining the value of the dual is the main purpose, in an effort to obtain a lower bound of the primal, all these methods provide meaningful results. Notably, the proposed scheme displays a finite termination property and more rapid convergence on the problems tested. With the same diminishing steplength rules, steepest ascent direction performs almost the same as arbitrary sub-gradient direction in terms of the value. The dual optimal values from the proposed method may not be monotonically increasing for two reasons: first, the proposed method adopts backtracking; second, the proposed method may jump from one face of the dual function to the other one, these jumps are kept as long as γ decreases.

Since the 3-generator problems are relatively straightforward, it can be independently verified that such problems admit unique solutions. In Figure 3, we plot the norm of the Euclidean distance from these solutions. It is seen that the general methods display poor local convergence behavior and premature termination may lead to significant error, as a consequence of the flatness of the value function around the optimum. Consequently, a small change in value may correspond to a huge difference in the solution. This fact also suggests that the methods effective for obtaining good

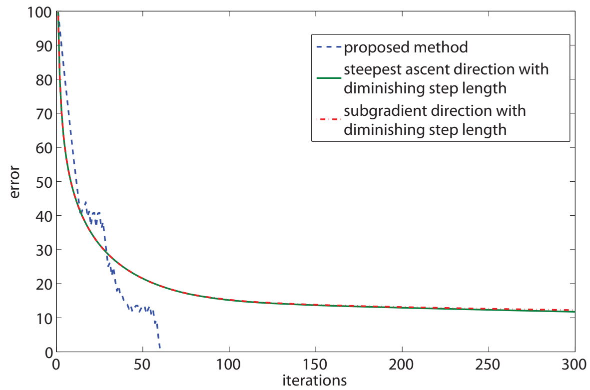


Fig. 3. Price errors through iterations

approximations may not always be suitable for the convex hull pricing problem.

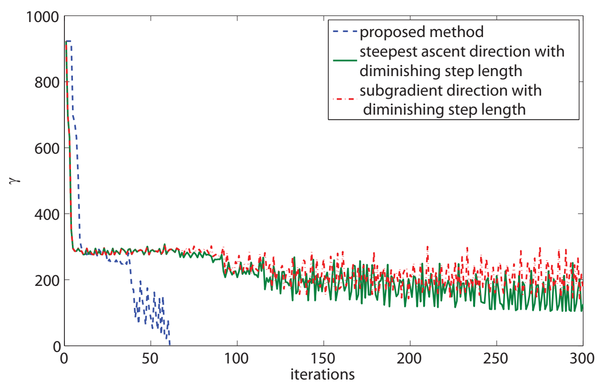


Fig. 4. Errors in terms of γ through iterations

To illustrate why the proposed methods are more efficient, we plot γ, the Euclidean distance between the convex hull of aggregated auction-surplus-maximization quantity set and demand, in Figure 4. We can see that all the methods display a non-monotone property while the proposed method corrects this through backtracking. Indeed, all points that lead to increased γ function are discarded in the proposed method. Furthermore, the lower envelope of the plot is representative of the *accepted iterates* for the proposed method. When only the accepted iterates are considered (post backtracking), the scheme does display a monotonically decreasing behavior in error. In contrast, the traditional methods make little progress after 100 iterations. We also note under the same diminishing steplength rules, the steepest direction is slightly better than the arbitrary sub-gradient direction, which implies a small number of iterations hit the null measure non-differentiable points. This can be explained by the fact that the "zigzagging" is generally around the non-differentiable points. With a diminishing steplength rule, the general methods are more and more likely to arrive these points. Such observations can be viewed as a numerical evidence of the convergence behavior of these general methods, but the required number of iterations makes them less practical.

	3-unit	10-unit	26-unit
number of total iterations	61	236	102
number of backtracking iterations	26	181	67
CPU time (s)	73.1	990.2	1057.4

TABLE II
SCALABILITY OF THE PROPOSED METHOD TO POWER SYSTEMS SIZE

Numerical tests are also carried on the widely-used IEEE 10-unit and 26-unit systems, which include all standard constraints of unit commitment problem. Necessary technical data are given in [19] and [20], respectively. We approximate the quadratic cost curves in the original data by 4-segment piecewise linear offer functions.

The scalability of the method is numerically demonstrated by the comparison of the results of the 3-unit, 10-unit, and 26-unit testing systems, as shown in Table II. We observe that the CPU time is roughly proportional to the number of units times the number of iterations. This is because most of the CPU time is spent on solving the optimization problem (7) and this problem is solved for every unit during every iteration. We also observe that the number of iterations does not grow significantly with the size of problem. The 3-unit system requires the least number of iterations, and a smaller fraction of backtracking iterations. This is because the generators in this system are less constrained. In fact, there is no intertemporal constraint in the 3-unit system, making the hourly prices independent of each other. For more realistic systems, such as the 10-unit and 26-unit systems, the commitment decisions are strongly coupled from a temporal standpoint. Therefore, both the total number of iterations and the fraction of backtracking iterations increase. Another interesting observation is that the 26-unit system actually converges faster than the 10-unit one. While appearing counter intuitive, this phenomenon can be explained by the fact that 26-unit system has a wider generation technology mix, leading to a wider offer price range, than the 10-unit system. Consequently, a lot of generators either price themselves out of the market or operate as base units, despite changing prices. In contrast, the offer prices of each generator in the 10-unit system are much closer to each other, which makes the system far more sensitive to the prices changes across the iterations.

The performance of the method under different steplength, as well as the robustness of the proposed method with respect to the constant steplength rule, is tested on the 26-unit system. In Figure 5, we plot the number of iterations needed to reach the optimum with maximum steplength choices, denoted by c, as 5, 10, 15, and 20. For each choice, three bars represent the numbers of iterations for demand scaled to 95%, 100%, and 105% of the original data, where the lower part of each bar represents the numbers of backtracking. As the figure shows, at least two thirds of the iterations represent backtracking: these iterates, although discarded, provide valuable information for the succeeding iterations. Furthermore, the proposed method shows a relatively consistent performance for different demand profiles. On the other hand, the method is more sensitive to the selection of c: larger steplength appear to require more number of iterations to converge. In fact, more aggressive steplengths

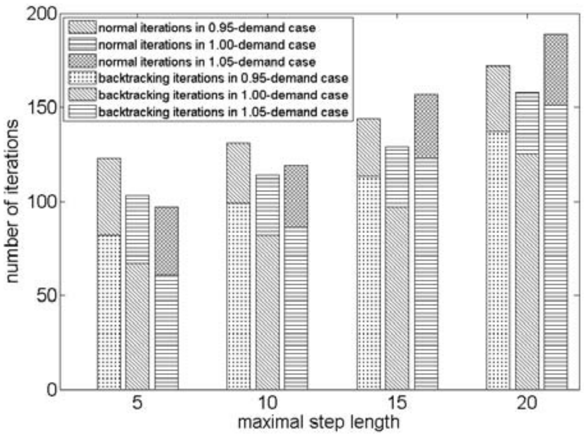


Fig. 5. Number of iterations for different demand with different maximal steplength

may move the iterates faster towards to the optimum, but may also introduce a lot of overshooting that needs to be corrected by backtracking. In reality, due to the strong daily, weekly and seasonal pattern of the markets, we may have a relatively good initial guess of the prices. In this case, a smaller steplength to reduce overshooting may be a better choice. Of course, other step length rules for the maximal steplength selection, such as diminishing steplength, can also be incorporated into the proposed method.

VI. Conclusion

Pricing is a key component of any electricity market design. Before making any changes to pricing schemes, extensive analysis, simulation and experiments, are necessary to avoid possibly undesired effects. This paper provides a computational tool to study convex hull prices. Traditional computational methods for maximizing the Lagrangian dual, in general, are characterized by poor local convergence and therefore cannot meet the need of electricity markets to obtain prices in an efficient fashion. In contrast, the proposed method exploits the structural information of these markets, and achieves significant gains in performance.

References

[1] G. J. Stigler, *The theory of price*, 4th ed. New York: Macmillan Publishing Company, 1987.

[2] R. Johnson, S. Oren, and A. Svoboda, "Equity and efficiency of unit commitment in competitive electricity markets," *Utilities Policy*, vol. 6, no. 1, pp. 9–19, 1997.

[3] R. Sioshansi, R. O'Neill, and S. Oren, "Economic consequences of alternative solution methods for centralized unit commitment in day-ahead electricity markets," *IEEE Transactions on Power Systems*, vol. 23, no. 2, pp. 344–352, 2008.

[4] W. Hogan and B. Ring, "On minimum-uplift pricing for electricity markets," *Electricity Policy Group*, 2003.

[5] P. Gribik, W. Hogan, and S. Pope, "Market-clearing electricity prices and energy uplift," Tech. Rep., 2007.

[6] M. Fisher, "The lagrangian relaxation method for solving integer programming problems," *Management Science*, vol. 50, no. 12, pp. 1861–1871, 2004.

[7] Q. Zhai, X. Guan, and J. Cui, "Unit commitment with identical units successive subproblem solving method based on lagrangian relaxation," *Power Systems, IEEE Transactions on*, vol. 17, no. 4, pp. 1250–1257, 2002.

[8] J. Goffin and J. Vial, "Convex nondifferentiable optimization: A survey focused on the analytic center cutting plane method," *Optimization Methods and Software*, vol. 17, no. 5, pp. 805–867, 2002.

[9] M. Madrigal and V. H. Quintana, "An interior-point/cutting-plane method to solve unit commitment problems," *IEEE Transactions on Power Systems*, vol. 15, no. 3, pp. 1022–1027, 2000.

[10] A. Borghetti, A. Frangioni, F. Lacalandra, and C. Nucci, "Lagrangian heuristics based on disaggregated bundle methods for hydrothermal unit commitment," *Power Systems, IEEE Transactions on*, vol. 18, no. 1, pp. 313–323, 2003.

[11] X. Zhao, P. Luh, and J. Wang, "Surrogate gradient algorithm for lagrangian relaxation," *Journal of Optimization Theory and Applications*, vol. 100, no. 3, pp. 699–712, 1999.

[12] C. Wang, P. B. Luh, P. Gribik, Z. Li, and P. Tengshun, "A subgradient-based cutting plane method to calculate convex hull market prices," in *Power & Energy Society General Meeting, 2009. PES '09. IEEE*, 2009.

[13] ——, "The subgradient-simplex based cutting plane method for convex hull pricing," in *Power & Energy Society General Meeting, 2010. PES '10. IEEE*, 2010.

[14] R. Rockafellar, *Convex analysis*. Princeton University Press, 1997.

[15] Q. Zhai, X. Guan, and J. Yang, "Fast unit commitment based on optimal linear approximation to nonlinear fuel cost: Error analysis and applications," *Electric Power Systems Research*, vol. 79, no. 11, pp. 1604–1613, 2009.

[16] IBM, "IBM ILOG CPLEX v12.1, user's manual," 2009.

[17] K. Kiwiel, *Methods of descent for nondifferentiable optimization*. New York: Springer-Verlag, 1985.

[18] N. Shor, *Minimization methods for non-differentiable functions*. New York: Springer-Verlag, 1985.

[19] S. Kazarlis, A. Bakirtzis, and V. Petridis, "A genetic algorithm solution to the unit commitment problem," *Power Systems, IEEE Transactions on*, vol. 11, no. 1, pp. 83 –92, Feb. 1996.

[20] C. Grigg, P. Wong, P. Albrecht, R. Allan, M. Bhavaraju, R. Billinton, Q. Chen, C. Fong, S. Haddad, and S. Kuruganty, "The ieee reliability test system-1996. a report prepared by the reliability test system task force of the application of probability methods subcommittee," *IEEE Transactions on Power Systems*, vol. 14, no. 3, pp. 1010–1020, 1999.

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