Piecewise Linear Test Functions for Stability and Instability of Queueing Networks*

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Abstract

We develop the use of piecewise linear test functions for the analysis of stability of multiclass queueing networks and their associated fluid limit models. It is found that if an associated LP admits a positive solution, then a Lyapunov function exists. This implies that the fluid limit model is stable and hence that the network model is positive Harris recurrent with a finite polynomial moment. Also, it is found that if a particular LP admits a solution, then the network model is transient.

Running head : Stability and Instability of Queueing Networks

Keywords : Multiclass queueing networks, ergodicity, stability, performance analysis.

1 Introduction

It has generally been taken for granted in queueing theory that stability of a network is guaranteed so long as the overall traffic intensity is less than unity and in recent years there has been much analysis which supports this belief for special classes of systems, such as single class queueing networks (see Borovkov [2], Sigman [47], Meyn and Down [38], Foss [19], and Foss and Baccelli [1]). This intuition was shown to be false in general by Kumar and Seidman [27], where it was demonstrated that for a multiclass system with an unintelligently chosen buffer priority scheduling policy, instability will occur even for loads less than unity. Similar counterexamples were reported in Lu and Kumar [32], and Rybko and Stolyar [44] (the latter for a stochastic network). The question of whether or not more natural policies would always be stable, such as the standard FIFO policy, remained open until the recent work of Bramson [4] and Seidman [46]. The re-entrant structure of these and all other counterexamples so far discovered is similar to the topology of many semiconductor manufacturing plants.

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A particular case of the stochastic Lyapunov function method is *Foster's criterion*. This technique has been successfully used to investigate the behavior of queues and queueing networks which admit a Markovian realization since Foster [20] in a wide variety of contexts. In particular, Botvich and Zamyatin [3], Fayolle et. al. [17, 18], Malyšhev et. al. [33, 35, 34], Men'šikov [36], and others have obtained a series of results which have shed much light on the stability of random walks and queueing networks on low dimensional spaces and much of this work is based upon stochastic Lyapunov techniques for Markov chains, in the spirit of Kingman [24]. A survey of recent methods for determining whether a given network or scheduling policy is stable is presented in Down and Meyn [15].

Recently, Kumar and Meyn [26] have developed a method for automating the stability analysis of multiclass networks. A class of quadratic test functions is used and it is found that the search for a candidate function that satisfies Foster's criterion may be posed as a linear programming problem. Various network examples are studied and while the method is effective, the algorithm developed has difficulty definitively deducing stability regions for some networks.

Botvich and Zamyatin [3] studied the stability of a particular network (which was introduced in [44]) using piecewise linear functions. They were able to completely classify stability for the network, which was not possible using quadratic test functions. In work concurrent to that presented here, Dai and Weiss in [12] have applied piecewise linear Lyapunov functions directly on multiclass fluid limit models for the analysis of stability. Very recently, Dai [9] and Bertsimas et. al. [6] have developed specialized methods to address the stability of fluid network models, which have been shown to be exact for two station networks.

In this paper, we present an approach that uses the LP approach of [26] and is inspired by the treatment of the example in [3]. We find that a suitable test function may be found by first devising a linear program to construct a suitable piecewise linear function and then by smoothing the resulting function, we obtain a Lyapunov function. For exponential networks this directly implies V-uniform ergodicity for the model, which implies that arbitrary polynomial moments converge exponentially quickly to their steady state values. In the general renewal case, we apply recent results in Dai [8], and Dai and Meyn [10] to find that the model is positive recurrent with finite mean queue lengths in steady state. Recently, it has been shown by Meyn [37] that instability of an associated fluid limit model implies transience of the network model. It is found that if the fluid limit model evolves on a reduced state space, piecewise linear test functions for instability may be constructed. This combination of stability and instability tests suggests that the methods developed in this paper may be appropriate for stability analysis in general. Some of the work in this paper was presented in abridged form in [14, 13].

We begin with a description of the models which we treat.

2 The Re-Entrant Line

For simplicity, in this paper we consider exclusively *re-entrant lines*, which are a particular example of the multiclass networks with feedback described by Harrison

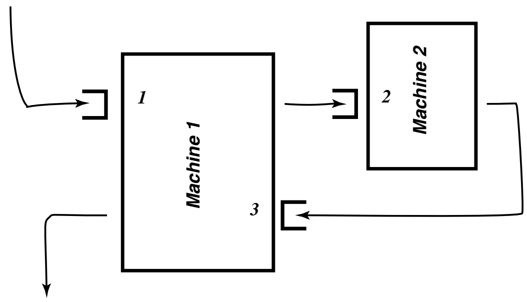

Figure 1. A Re-entrant network model

and Nguyen in [21]. A simple example of a re-entrant line is the network described in Figure 1 consisting of two machines and three buffers, with a single server at each machine. A crucial feature of this model is the re-entrant structure, which is typical of semiconductor manufacturing models (see Kumar [25]). The extension of the results presented in this paper to general multiclass queueing networks is straightforward.

The models which we consider consist of S single server stations, and a total of K buffers, which are located among the server stations. In the example illustrated in Figure 1, $K = 3$ and $S = 2$. Customers at buffer k are serviced at station $\sigma(k) \in \{1, \dots, S\}$, with service times $\{\eta_k(n), n \geq 1\}$. The description of the network is simplified since routing is deterministic: customers arrive to buffer one, with inter-arrival times $\{\xi(n), n \geq 1\}$, where they wait in queue until service. After a service of length $\eta_1(n)$ is completed, a customer moves on to buffer two, and so forth, until it finally reaches buffer K. After service is completed at this final queue, the customer leaves the network.

It is assumed that service disciplines are "state dependent", non-idling, and pre-emptive. Hence, after a service is completed at a server station, a new buffer is chosen for service at this station based solely upon the current buffer levels in the system. Throughout this paper we make the following three assumptions on the network.

(A1) $\xi, \eta_1, \dots, \eta_K$ are mutually independent and i.i.d. sequences.

(A2) For some integer $p \geq 1$,

$$\mathsf{E}[|\eta_k(1)|^p] < \infty \text{ for } k = 1, \dots, K, \quad \text{and} \quad \mathsf{E}[|\xi(1)|^p] < \infty.$$

(A3) For some positive function p on $\mathbb{R}_+$ and some integer j_0,

$$\mathsf{P}(\xi(1) \geq x) \quad > \quad 0 \quad \text{for all} \quad x > 0.$$

$$\mathsf{P}(\xi(1) + \dots + \xi(j_0) \in dx) \quad \geq \quad p(x)\, dx \quad \text{and} \quad \int_0^\infty p(x)\, dx > 0.$$

Occasionally, we will strengthen conditions (A2), (A3) to

(A4) The random variables $\xi, \eta_1, \dots, \eta_K$ are exponentially distributed.

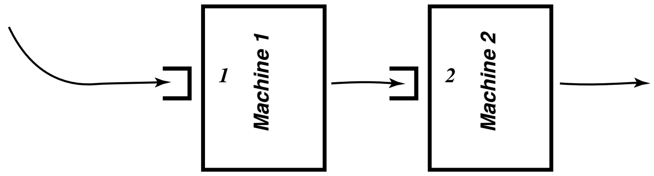

Figure 2. Two queues in tandem

For notational convenience set $\mu_k = 1/\mathsf{E}[\eta_k(1)]$ and $\lambda = 1/\mathsf{E}[\xi(1)]$.

For each $\sigma = 1, \ldots, S$ we let $C_\sigma = \{k : \sigma(k) = \sigma\}$ and define the *nominal load* at station σ as $\rho_\sigma = \sum_{k\in C_\sigma} \lambda/\mu_k$. To guarantee any practical form of stability for the network it is obviously necessary that the capacity constraint $\rho_\sigma < 1$ be satisfied for each $\sigma = 1, \ldots, S$. However, as cited in the introduction, numerous examples have been constructed which show that this condition is not sufficient for stability.

We define a Markov process $\boldsymbol{\Phi} = \{\Phi_t : t \geq 0\}$ which describes the dynamics of the queueing network. We will let P_x denote the probability distribution and E_x the corresponding expectation operator, under the initial condition $\Phi_0 = x$. For each t, $\Phi_t = (X_1^\top(t), \ldots, X_K^\top(t))^\top$, where $X_i(t)$ summarizes all the network information at buffer i. For the rest of this section we will assume that the scheduling policies are of the buffer priority type. This is simply for ease of exposition. For examples of construction of Markovian representations for other scheduling policies, see [8]. Under (A1) and (A4) one may take $X_i(t) = b_i(t) =$ the queue length at buffer i, so that $\boldsymbol{\Phi}$ is a countable state space Markov chain with state space $\mathsf{X} = \mathbb{Z}_+^K$. However, if the distributions are general, then it is necessary to augment this state description with the residual exogenous interarrival times and residual service times to preserve the Markov property. At time t, let $u(t)$ be the residual exogenous interarrival time, and $v_k(t)$ be the residual service time at buffer k. These residual processes are taken to be right continuous. For the state process $\boldsymbol{\Phi}$, we then take $X_1(t) = (u(t), b_1(t), v_1(t))^\top$ and $X_i(t) = (b_i(t), v_i(t))^\top$ for $i \geq 2$. The state space X in this case will be equal to some subset of Euclidean space $\mathbb{R}^{2K+1}$ and will not be countable in this more general framework.

For example, take the tandem queue illustrated in Figure 2. For this model, the state process $\boldsymbol{\Phi}$ becomes

$$\Phi_t = (u(t), b_1(t), v_1(t), b_2(t), v_2(t))^\top \tag{1}$$

and the state space X is evidently equal to a subset of $\mathbb{R}_+^5$.

Under the assumptions (A1)-(A3), the state process $\boldsymbol{\Phi}$ has several desirable topological as well as measure-theoretic structural properties. As argued in [8], the process $\boldsymbol{\Phi}$ is a strong Markov process. The following theorem will simplify several of the results presented below. For a proof and definitions of the terms, see [38] and [40].

Theorem 2.1 *Suppose that assumptions (A1)-(A3) hold. Then all compact sets are petite for some skeleton chain* $\boldsymbol{\Phi}^\delta = \{\Phi_0, \Phi_\delta, \Phi_{2\delta}, \ldots\}$, *and hence* $\boldsymbol{\Phi}$ *is a* ψ*-irreducible and aperiodic T-process.* □

We begin by describing recent approaches to stability for the network model with state process $\boldsymbol{\Phi}$.

3 Criteria For Stability

Many of the methods for addressing stability which we describe in this paper are generalizations of Lyapunov's second method as described in, for example, LaSalle and Lefshetz [31]. We begin with a description of the stochastic Lyapunov technique, which is commonly known as Foster's Criterion, although it was developed for diffusions in the Russian literature prior to Foster (see Khas'minskii [23]). In this section, the assumption made on the scheduling policies is that they are non-idling and state dependent.

Stochastic Lyapunov functions To describe the stochastic Lyapunov function approach for continuous time models, we first define a version of the generator for the process $\boldsymbol{\Phi}$. We denote by $D(\widetilde{\mathcal{A}})$ the set of all functions $f\colon \mathsf{X} \to \mathbb{R}$ for which there exists a measurable function $g\colon \mathsf{X} \to \mathbb{R}$ such that for each $x \in \mathsf{X}$, $t > 0$,

$$\mathsf{E}_x[f(\Phi_t)] = f(x) + \mathsf{E}_x\Big[\int_0^t g(\Phi_s)\,ds\Big] \tag{2}$$

$$\int_0^t \mathsf{E}_x[|g(\Phi_s)|]\,ds < \infty. \tag{3}$$

We write $\widetilde{\mathcal{A}}f := g$ and call $\widetilde{\mathcal{A}}$ the *extended generator* of $\boldsymbol{\Phi}$; $D(\widetilde{\mathcal{A}})$ is called the domain of $\widetilde{\mathcal{A}}$. In many cases we can write

$$\widetilde{\mathcal{A}}V(x) = \lim_{h\downarrow 0} \frac{\int P^h(x,dy)V(y) - V(x)}{h} \tag{4}$$

where P^h is the transition semigroup for $\boldsymbol{\Phi}$.

In the following result we show if $V \in D(\widetilde{\mathcal{A}})$ is a positive function on X, and if the *drift* $\widetilde{\mathcal{A}}V\,(x)$ is suitably negative off some compact subset of X, then the network model will be positive recurrent. The proof follows from Theorem 6 of Meyn and Tweedie [42] and Theorem 2.1.

Theorem 3.1 *For the Markovian network with state process $\boldsymbol{\Phi}$ satisfying (A1) - (A3), suppose that there exists some compact set $C \subset \mathsf{X}$, some $\gamma > 0$, and some non-negative function V such that*

$$\widetilde{\mathcal{A}}V(x) \le -\gamma + b\mathbb{1}_C(x). \tag{5}$$

Then $\boldsymbol{\Phi}$ is positive Harris recurrent. □

While this result holds for a vast class of models – it is only ψ-irreducibility that is really required – because of the simpler state description, in this section we will restrict our attention to exponential network models. This will be relaxed below.

Even though for network models the operator $\widetilde{\mathcal{A}}$ is rarely a differential operator, calculus plays an important role in analyzing the drift $\widetilde{\mathcal{A}}V$. This is because typically the function V will be a norm, or a polynomial function of a norm, on the state space X and we will see that we can typically assume that V can be extended to form a smooth (C^∞) function on a suitable subset of Euclidean space. For instance, in [38] a linear function is constructed and Dupuis and Williams [16] explicitly construct such a test function for a class of reflected Brownian motions. Theorem 3.7, which forms a large part of the foundation of this paper, shows how a smooth function may be constructed given a piecewise linear test function.

Consider the exponential model satisfying (A1) and (A4) with $\mathsf{X} = \mathbb{Z}_+^K$ and define the *mean velocity vector* $\Delta(x) \in \mathbb{R}^K$ for $x \in \mathsf{X}$ so that

$$\mathsf{E}_x[x(h)] - x = \Delta(x)h + o(h),$$

where $x = (x_1, \ldots, x_K)^\top$ is the vector of queue lengths. For the re-entrant line the mean velocity vector may be explicitly calculated as

$$\Delta(x) = \lambda e^1 + \sum_{k=1}^{K} [\mu_{k-1} w_{k-1}(x) - \mu_k w_k(x)] e^k \tag{6}$$

where e^k denotes the kth standard basis element in $\mathbb{R}^K$. For $k = 1, \ldots, K$ we define $w_k(x) = 1$ if buffer k is in service and zero otherwise; we set $w_0(x) \equiv 0$. More generally, we can allow processor sharing policies so that $w_k(x)$ takes on values between 0 and 1. By the non-idling constraint, the conditions on $\{w_k\}$ become

$$w_k(x) \geq 0, \qquad \text{and} \qquad \sum_{k \in C_\sigma} w_k(x) = 1 \qquad \text{whenever} \sum_{k \in C_\sigma} x_k > 0. \tag{7}$$

In the following result we perform a Taylor series expansion to write

$$\widetilde{\mathcal{A}}V\,(x) = \Delta(x)^\top V'(x) + o(\widetilde{\mathcal{A}}V\,(x)) \tag{8}$$

where V' denotes the gradient of V and $o(\widetilde{\mathcal{A}}V\,(x)) \ll |\widetilde{\mathcal{A}}V\,(x)|$ for large x. This gives rise to a geometric approach to stability, which was first described in this operations research context by Kingman in [24].

Theorem 3.2 *Consider the network model satisfying Assumptions (A1) and (A4). Suppose that* $V\colon \mathbb{R}^K \to \mathbb{R}$ *is a* C^2 *function and that for some* $\gamma_0 > 0$*,*

$$\Delta(x)^\top V'(x) < -\gamma_0 \qquad x \neq 0. \tag{9}$$

Suppose further that $V''(x) \to 0$ *as* $x \to \infty$*. Then (5) holds for some* $\gamma > 0$*, so that the network model is positive Harris recurrent.*

Moreover, under these conditions there exist constants δ, $\varepsilon > 0$, $B < \infty$*, such that*

$$|\mathsf{E}_x[f(\Phi_t)] - \int f\, d\pi| \leq BV_*(x)e^{-\varepsilon t}, \qquad t \geq 0, \tag{10}$$

for any function f *satisfying the bound* $|f| \leq V_*$*, where* $V_*(x) = \exp(\delta V(x))$ *and* π *is the invariant probability.*

Proof We have for the exponential network

$$\widetilde{\mathcal{A}}V(x) = \lambda[V(x+e^1) - V(x)] + \sum_{k=1}^{K} \mu_k w_k(x)[V(x+e^{k+1}-e^k) - V(x)]$$

where again e^i denotes the ith standard basis element in $\mathbb{R}^K$, we set $e^{K+1} = 0$, and as usual we set $w_0(x) \equiv 0$. It then follows from the Mean Value Theorem that (8) holds with $o(\widetilde{\mathcal{A}}V(x)) \ll 1$ for large x and this together with (6) shows that (5) does hold for a suitably large compact set.

To establish (10), set $V_*(x) = \exp(\delta V(x))$. For sufficiently small δ, we can perform the Taylor series approximation used in Theorem 16.3.1 of Meyn and Tweedie [39] to show that for some $\gamma_* > 0$, $b_* < \infty$, and a compact set C,

$$\widetilde{\mathcal{A}}V_*(x) \le -\gamma_* V_*(x) + b_* \mathbb{1}_C(x).$$

The limit (10) then follows from Theorem 6.1 of [41]. □

Fluid limit models and stability Chen and Mandelbaum in [5] show generally that complex network models can be approximated by a fluid limit model. This gives rise to a new approach to stability, which is far more general than the direct stochastic Lyapunov approach described above. By examining an essentially deterministic fluid limit model, we may drop the restrictive exponential assumption, while preserving the geometric approach to stability indicated in Theorem 3.2.

Let $\tau(t)$ denote the vector of cumulative busy times up to time t for the re-entrant line. Define $b^x(t) = \frac{1}{|x|}b(t|x|)$ and similarly, $\tau^x(t) = \frac{1}{|x|}\tau(t|x|)$. The *fluid limit model* is defined as the set of weak limits for (b^x, τ^x) as $|b(0)| \to \infty$, with $(u(0) + \sum v_i(0))/|b(0)| \to 0$ (see [10]). Any particular weak limit $(Q(t), T(t))$ will be called a *fluid limit*. For the re-entrant line, the fluid limits satisfy the following three conditions. It is stressed that the fluid limit model is not characterized by (F1–F3) and that frequently there is not a unique solution for a given initial condition. Further constraints are imposed on specific network examples given below. Although uniqueness of the solution is still not guaranteed, this additional structure is essential when attempting to verify stability or instability of the fluid limits.

(F1) *Queue length evolution*: $Q(t) = \lambda t e^1 + \sum_{k=1}^{K} [\mu_{k-1}T_{k-1}(t) - \mu_k T_k(t)]e^k$;

(F2) *Busy time evolution*: $T_0(t) \equiv 0$ and for any $k = 1, \dots, K$,

$$T_k(0) = 0 \text{ and } T_k \text{ is increasing and continuous;}$$

(F3) *Non-idling constraint*: For any $\sigma = 1, \dots, S$, with $B_\sigma(t) = \sum_{k \in C_\sigma} T_k(t)$, for almost all t,

$$\tfrac{d}{dt}B_\sigma(t) = 1, \text{ whenever } \textstyle\sum_{k \in C_\sigma} Q_k(t) > 0.$$

The significance of the fluid limit model lies largely in the following result, which is taken from [8, 10]:

Theorem 3.3 *Suppose that the fluid limit model is asymptotically stable so that $Q(t) \to 0$ as $t \to \infty$, for any initial condition $Q(0) \in \mathbb{R}^K$. Then, under Assumptions (A1–A3) the network with state process $\boldsymbol{\Phi}$ is positive Harris recurrent and for each initial condition*

$$\lim_{t\to\infty} \mathsf{E}_x[b_i(t)^{m-1}] = \mathsf{E}_\pi[b_i(0)^{m-1}] < \infty \qquad 1 \le i \le K,\ 1 \le m \le p-1 \tag{11}$$

where π is the invariant probability and p is the integer used in (A2).

Theorem 3.3 raises the question, *when will the fluid limit model be asymptotically stable*? This can be addressed using a Lyapunov approach, analogous to Theorem 3.2. We first define an analogue of the vector $\Delta(x)$ for the exponential network. For each $k = 1, \dots, K$ set $W_k(t) := \frac{d}{dt} T_k(t)$ whenever the derivative exists and define the *velocity vector* $\delta(t)$ for the fluid limit model as

$$\delta(t) = \frac{d}{dt} Q(t) = \lambda e^1 + \sum_{k=1}^{K} [\mu_{k-1} W_{k-1}(t) - \mu_k W_k(t)] e^k.$$

By condition (F3) we have the analogue of (7)

$$W_k(t) > 0, \quad \text{and} \quad \sum_{k\in C_\sigma} W_k(t) = 1 \quad \text{whenever} \sum_{k\in C_\sigma} Q_k(t) > 0. \tag{12}$$

The velocity vector $\delta(t)$ is multivalued in general and is dependent on the position $Q(t)$. Let $\delta(x)$ denote the set of all possible velocity vectors over all fluid limits $Q(t)$, with $Q(t) = x$.

Theorem 3.4 *Suppose that $V\colon \mathbb{R}^K \to \mathbb{R}$ is a C^1 function and that for some $\gamma_0 > 0$, the fluid limit model satisfies*

$$\delta^\top V'(x) < -\gamma_0 \qquad \delta \in \delta(x), x \ne 0. \tag{13}$$

Then the fluid limit model is asymptotically stable.

Proof The function $V(Q(t))$ is absolutely continuous with

$$V(Q(t)) = V(Q(0)) + \int_0^t V'(Q(r))^\top \delta(r)\, dr. \tag{14}$$

Then clearly, from (13), $Q(t) = 0$ for all $t \ge V(Q(0))/\gamma_0$. □

Because the specific form of $\delta(x)$ cannot always be computed exactly, we will typically use only the constraints (12) and additional constraints imposed by the scheduling policy in checking the drift criterion (13). In this case, the hypotheses of Theorem 3.4 and Theorem 3.2 are nearly identical. Theorem 3.4 requires fewer assumptions since the bounds on V'' and Assumption (A4) are not needed. However, Theorem 3.2 has a somewhat stronger conclusion due to the geometric convergence (10).

To complete the picture, we need an analogous test for instability. This may allow us to completely classify queueing networks, as finding a Lyapunov function of a specific form is of course only a sufficient condition for stability. The required instability result is found in the recent work [37].

Theorem 3.5 *Suppose that the fluid limit model is unstable in the sense that for some* $\varepsilon_0 > 0$,

$$|Q(T)| \geq \varepsilon_0 T, \qquad T \geq 0, \tag{15}$$

for all initial conditions $Q(0)$, *with* $|Q(0)| = 1$. *Then, for any* $1 \geq q > 0$, *there exists* $B < \infty$ *such that whenever* $|x| \geq B$,

$$\mathsf{P}_x\{\boldsymbol{\Phi} \to \infty\} \geq q.$$

□

This gives rise to the following Lyapunov based approach to test for instability.

Theorem 3.6 *Suppose that* $V\colon \mathbb{R}^K \to \mathbb{R}$ *is a* C^1 *norm on* X *and that for some* $\gamma_0 > 0$, *the fluid limit model satisfies*

$$\delta^{\top} V'(x) > \gamma_0, \qquad \delta \in \delta(x), x \neq 0. \tag{16}$$

Then the fluid limit model is unstable in the sense of (15).

Proof The result is obvious using (14). □

Now that we have developed a Lyapunov approach to stability, we will describe specific classes of functions for use in analyzing network models.

Piecewise linear test functions In the remainder of this section we will consider a simple example to show how to construct suitable test functions. For convenience, we restrict attention to the mean velocity vector field and the generator-based drift criterion. The same development goes through for the fluid model by considering its associated velocity vector field.

Consider the tandem queue illustrated in Figure 2. For this model the mean velocity vector can be expressed

$$\Delta(x) = \begin{pmatrix} \lambda - \mu_1 \mathbb{1}(x_1 > 0) \\ \mu_1 \mathbb{1}(x_1 > 0) - \mu_2 \mathbb{1}(x_2 > 0) \end{pmatrix} \tag{17}$$

In Figure 3 we have illustrated the mean velocity vector $\Delta(x)$ defined in (6) in the special case $\lambda = 1$, $\mu_1 = \mu_2 = 1.1$. These vectors have been skewed to exaggerate their relative angles.

To address stability we first consider a linear function $V(x) = x^{\top} c$ where $c \in \mathbb{R}^K$ is chosen so that $V(x) \geq 0$ for all $x \in \mathbb{R}^K_+$. To satisfy (5), the entire vector $\Delta(x_0)$ with origin at x_0 must lie within the half space $\{x \in \mathbb{R} : c^{\top} x \leq c^{\top} x_0\}$ for all $x_0 \in \mathsf{X}$. It is clear from Figure 3 that this is impossible in this example for any fixed c.

Quadratics are effective in determining stability for the tandem queue. In this case for all $\rho_1, \rho_2 < 1$, a quadratic Lyapunov function may be determined, as in Fayolle [17], and Down and Meyn [15]. The one difficulty with this approach is that the functions themselves depend on the parameters of the network in a complex way.

We now search for a piecewise linear function which satisfies the geometric condition (9), where $V'(x)$ now denotes any subgradient vector at x. We will take V of the specific form

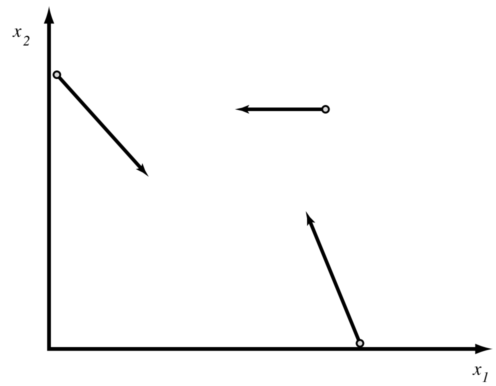

Figure 3. The mean velocity vector field $\Delta(x)$ for the tandem queue.

$$V_d(x) = \max\{x^\top V^i : 1 \le i \le m\} \tag{18}$$

where $V^i \in \mathbb{R}_+^K$. The conditions on V which we will impose are described as follows. Extend the definition of the mean velocity vector $\Delta(x)$ given in (6) to all of $\mathbb{R}_+^K$ by only requiring that (7) holds. Let d be the vector $({V^1}^\top, \ldots, {V^m}^\top)^\top$, let $J(x,d)$ denote those indices j such that $V(x) = x^\top V^j$, and suppose that for some $\gamma_0 > 0$,

$$\Delta(x)^\top V^j \le -\gamma_0, \qquad j \in J(x,d) \tag{19}$$

for *any* mean velocity vector $\Delta(x)$ which satisfies the condition (7), and all $x \neq 0$.

We do not know if the function V_d then satisfies (5), however the following result shows that this is irrelevant:

Theorem 3.7 *Suppose that a function V_d of the form (18) exists, satisfying (19). Then there exists a smooth function $V_s\colon \mathbb{R}^K \to \mathbb{R}_+$ and $\gamma > 0$ such that $\Delta(x)^\top V_s'(x) \le -\gamma$ for all mean velocity vectors $\Delta(x)$ satisfying (7).*

The function V_s is radially homogeneous in the sense that

$$V(x) = (1/\alpha)V(\alpha x) \qquad x \in \mathbb{R}_+^K, \quad \alpha > 0, \tag{20}$$

and hence also $V''(x) \to 0$ as $x \to \infty$.

Proof First we show that if (19) holds for some x_0 and some vector $d = ({V^1}^\top, \ldots, {V^m}^\top)^\top$, then there exists an $\varepsilon > 0$ such that

$$\Delta(x_0)^\top z^j \le -\tfrac{1}{2}\gamma_0 \qquad \text{for all } z \in B_\varepsilon(d),\ j \in J(x_0, z), \tag{21}$$

whenever $\Delta(x_0)$ satisfies (7). Here $B_\varepsilon(d)$ denotes the open ball $\{z : |z-d| < \varepsilon\}$. By (19) and continuity in d, it is clear that we may choose $\varepsilon_0 > 0$ so small that $\Delta(x_0)^\top z^j \le -\frac{1}{2}\gamma_0$ for $j \in J(x_0,d)$ if $|d-z| < \varepsilon_0$. We also have by continuity that $J(x_0,z) \subseteq J(x_0,d)$ for all z with $|z-d|$ sufficiently small, say less than ε_1. Thus (21) holds with $\varepsilon = \min(\varepsilon_0, \varepsilon_1)$.

The fact that we are searching among all non-idling policies yields radial homogeneity of the model. By a compactness argument, we may find a fixed $\varepsilon > 0$ so that (21) holds for any x_0 and any admissible mean velocity vector $\Delta(x_0)$. Let $p(\,\cdot\,)$ be a C^∞ probability density supported on the open ball $B_\varepsilon(d)$, and define

$$V_s(x) = \int V_z(x) p(z)\, dz.$$

This smoothing is as performed in [16]. The function V_s is C^∞, radially homogeneous, and satisfies $\Delta^\top(x) V_s'(x) \le -\frac{1}{2}\gamma_0$ for all $x \neq 0$. □

Consider now the tandem pair of queues under (A4) with state space $\mathbb{Z}_+^2$. By examining the mean velocity vectors $\Delta(x)$ for the model illustrated in Figure 3 one can see that the choice $x^\top V^1 = 2x_1$, $x^\top V^2 = x_1 + x_2$ gives rise to a function which satisfies the conditions of Theorem 3.7. To see that the inequality (19) holds in general under the capacity constraints, observe that for $x \neq 0$,

$$\Delta(x)^\top V^1 = -2(\mu_1 \mathbb{1}(x_1 > 0) - \lambda)$$

which is negative when $x_1 > 0$; and

$$\Delta(x)^\top V^2 = -(\mu_2 \mathbb{1}(x_2 > 0) - \lambda)$$

which is negative when $x_2 > 0$. It follows that (19) does hold and by Theorem 3.7 we again see that the network is positive Harris recurrent.

We see that in this example a single piecewise linear function may be used to establish the entire stability region. It is not possible to find a single quadratic function: for certain relative service rates, any valid quadratic test function becomes nearly singular as $\rho \uparrow 1$. We now give an approach to construct piecewise linear functions for complex network models.

4 Linear programming characterization

Theorem 3.7 indicates that a smooth Lyapunov function may be constructed based on a piecewise linear function. In this section we show that the search for a suitable piecewise linear function can be computer automated. This is non-trivial since the conditions in (19) impose highly nonlinear constraints on the components of V^j.

Non-idling policies In this section, we examine functions of the vector of buffer levels, $b(t)$. Note that $b(t)$ need not be the "state" of the system. It is only assumed that a Markovian state for the system may be constructed.

We first show how a piecewise linear function may be found when the only restriction on the scheduling policy is that it is non-idling. Let $b(t) = x$ and

$$V_\sigma(x) = \sum_{j=1}^{K} a_{\sigma,j} x_j,$$

where $a_{\sigma,j} \ge 0$. We also write $V_\sigma(x) = x^\top V^\sigma$, where $V^\sigma = (a_{\sigma,1}, \dots, a_{\sigma,K})^\top$.

We define the face $\Lambda_\sigma \subset \mathbb{R}_+^K$ as the set corresponding to the event that machine σ is empty:

$$\Lambda_\sigma = \{x \in \mathbb{R}_+^K : \sum_{i \in C_\sigma} x_i = 0\}.$$

Suppose we can find $V_\sigma, \sigma = 1, \dots, S$, such that the following two conditions hold:

$$\Delta(x)^\top V^\sigma \quad \leq \quad -1, \qquad \forall x \in \Lambda_\sigma^c \tag{22}$$

$$V_\sigma(x) \quad < \quad \max_{\sigma' \neq \sigma} V_{\sigma'}(x) \qquad \forall x \in \Lambda_\sigma, \ x \neq 0. \tag{23}$$

The bound (22) indicates that when there are customers at a machine, its associated linear function exhibits a negative drift. The dominance condition (23) implies that at each x, $V(x) = \max_\sigma V_\sigma(x)$ does not take on the value of a linear function which may exhibit a positive drift. It is clear that if the above conditions can be satisfied, the conditions of Theorem 3.7 hold for the piecewise linear function $V(x) = \max_\sigma V_\sigma(x)$.

The conditions in (22) may be regarded as linear constraints on the components of the vector V^σ. The constraint (23) is far from linear, but we may obtain a general sufficient condition that is linear using the following result:

Lemma 4.1 *Consider the linear functions*

$$d_i(x) \quad = \quad \sum_{j=1}^{k} a_{ij} x_j, \qquad \textit{for } i = 1, \dots, m$$

$$c(x) \quad = \quad \sum_{j=1}^{k} c_j x_j$$

where $\{a_{ij}, c_j : 1 \leq i \leq m, \ 1 \leq j \leq k\}$ *are real numbers. Assume that for all* j,

$$\frac{1}{m} \sum_{i=1}^{m} a_{ij} > c_j. \tag{24}$$

Then for all $x \in \mathbb{R}_+^k$, $x \neq 0$,

$$\max_{1 \leq i \leq m} (d_i(x)) > c(x).$$

Proof Let $d^i = (a_{i1}, \dots, a_{ik})^\top$, $c = (c_1, \dots, c_k)^\top$, so that condition (24) may be written

$$\frac{1}{m} \sum_{i=1}^{m} d^i > c,$$

where vector inequalities are interpreted componentwise. If $x \in \mathbb{R}_+^k$, $x \neq 0$, then

$$\max_{1 \leq i \leq m} (d_i(x)) \geq \frac{1}{m} \sum_{i=1}^{m} x^\top d^i > x^\top c = c(x).$$

□

It is important to stress here that (24) is indeed only a sufficient condition. One example which demonstrates that Lemma 4.1 is not necessary is given as follows:

$$d_1(x) = 1.1x_1, \qquad d_2(x) = 1.1x_2, \qquad c(x) = .9x_1 + .1x_2.$$

We will comment further on this result after presenting how $\{V_\sigma\}$ in (22) - (23) may be constructed using a linear program.

Theorem 4.2 *Consider the following linear program:*

$$\max \gamma$$

subject to the constraints:

$$\begin{array}{llcll}
(a1) & \gamma & \leq & 1 & \\
(a2) & \lambda a_{\sigma,1} + r_\sigma + \sum_{\{\sigma':\sigma'\neq\sigma\}} s_{\sigma,\sigma'} + \gamma \leq 0 & & & \text{for all } \sigma \\
(a3) & r_\sigma & \geq & \mu_j(a_{\sigma,j+1} - a_{\sigma,j}) & \text{for all } j,\ \sigma : j \in C_\sigma \\
(a4) & s_{\sigma,\sigma'} & \geq & \mu_j(a_{\sigma,j+1} - a_{\sigma,j}) & \text{for all } j,\ \sigma,\ \sigma' : j \in C_{\sigma'},\ \sigma' \neq \sigma \\
(a5) & s_{\sigma,\sigma'} & \geq & 0 & \text{for all } \sigma, \sigma' \\
(a6) & a_{\sigma,K+1} & = & 0 & \text{for all } \sigma \\
(a7) & a_{\sigma,j} & \geq & 0 & \text{for all } \sigma, j \\
(a8) & \frac{1}{S-1} \sum_{\sigma':\sigma'\neq\sigma} a_{\sigma',j} & \geq & a_{\sigma,j} + \gamma & \text{for all } \sigma,\ j : j \notin C_\sigma
\end{array}$$

If this LP has a solution $\gamma^* = 1$, then a piecewise linear function satisfying (22) - (23) exists.

If Assumptions (A1)–(A3) hold, then the network is positive Harris recurrent for all admissible non-idling policies, the queue length process $b_i(t)$ has a finite $(p-1)$-moment in steady state, and the limit (11) holds for each initial condition.

If in addition (A4) holds, then the network model is V_*-uniformly ergodic for all admissible non-idling policies, so that (10) holds with $V_*(x) = \exp(\delta V(x))$ for some $\delta > 0$ and all $|f| \leq V_*$.

Proof Note that if there is a solution to the LP with $\gamma^* < 1$, then by scaling the parameters $a_{\sigma,j}$ we can obtain a solution with $\gamma^* = 1$, so the only solutions are $\gamma^* = 0$ or 1.

Examining (22) for any σ, we see that upon applying (6)

$$\Delta(x)^\top V^\sigma \leq \lambda a_{\sigma,1} + \max_{j\in C_\sigma} \mu_j(a_{\sigma,j+1} - a_{\sigma,j}) + \sum_{\sigma':\sigma'\neq\sigma} [\max_{j\in C_{\sigma'}} \mu_j(a_{\sigma,j+1} - a_{\sigma,j})]^+. \quad (25)$$

Conditions (a2)–(a5) are (25), rewritten as linear constraints. Condition (a6) allows us to write (6) in a compact form, while (a7) ensures that for each σ, $V_\sigma(x) \geq 0$ for all $x \in \mathbb{R}_+^K$. Finally, (a8) expresses the dominance condition given in Lemma 4.1 which ensures that (23) holds.

The result follows on applying Theorem 3.7 and Theorem 3.2. □

The conditions of Lemma 4.1 may be reexamined with Theorem 4.2 in mind. Even though Lemma 4.1 gives only a sufficient condition under which the dominance condition (23) holds, the interaction between the desired bounds (22) and (23) is complex. The examples considered below suggest that for a stable network, there may be enough "slack" in the required drift conditions so that we may always choose the $\{a_{i,j}\}$ so that the sufficient condition of Lemma 4.1 is met.

It would also be of interest to determine whether it would be useful to generalize Lemma 4.1 by considering *weighted* averages in (24). This may provide more flexibility for the algorithm. Again it is an open question as to whether such added flexibility is needed, as the examples below suggest that the algorithm may actually characterize stability for re-entrant lines.

A three buffer system For our first example, we consider the two machine, three buffer system given in Figure 1. The only assumption we impose on the scheduling policy is that it is non-idling.

This system has been studied using quadratic Lyapunov functions in Section 9 of [26]. Let $\rho_1 = \lambda(1/\mu_1 + 1/\mu_3)$ and $\rho_2 = \lambda/\mu_2$. In [26], the case when $\mu_1 = \mu_3$ is considered and the set of pairs (ρ_1, ρ_2) for which a suitable quadratic can be found is determined numerically. It is found that a large amount of the stability region is captured, yet there is a region within $\rho_1, \rho_2 < 1$ where no conclusion about stability may be made.

Now, consider the approach of Theorem 4.2. We use two linear functions $V_1(x)$ and $V_2(x)$ and consider $V(x) = \max(V_1(x), V_2(x))$. The LP will not be included here, as it may be derived simply from Theorem 4.2. A similar approach has been used on the fluid limit model in [12].

A solution to the LP may be explicitly constructed when $\rho_1 < 1$ and $\rho_2 < 1$, which will show that for this example not only is this approach an improvement on quadratics but more importantly it completely characterizes the stability region. Consider the two functions consisting of the "work" (with a scaling factor) at machines 1 and 2, respectively.

$$\begin{aligned} V_1(x) &= \alpha((\frac{1}{\mu_1} + \frac{1}{\mu_3})x_1 + \frac{1}{\mu_3}(x_2 + x_3)) \\ V_2(x) &= \beta\frac{1}{\mu_2}(x_1 + x_2) \end{aligned}$$

It is straightforward to verify that whenever the capacity constraint $\rho_i < 1$ is met, the function V_1 has a negative drift when $x_1 + x_3 > 0$ and similarly, V_2 has a negative drift when $x_2 > 0$, which is (22). To satisfy (23), it is found that the following inequality must hold:

$$\frac{\mu_2}{\mu_3} < \frac{\beta}{\alpha} < \frac{\mu_2}{\mu_1} + \frac{\mu_2}{\mu_3}.$$

Trivially, this can be satisfied for all μ_1, μ_2, and μ_3.

The Dai-Wang example Consider the system given in Figure 4, with service rates $\mu_1 = 10$, $\mu_2 = 20$, $\mu_3 = 10/9$, $\mu_4 = 20$, and $\mu_5 = 5/4$. Also, define $\rho := \lambda/\mu_1 + \lambda/\mu_2 + \lambda/\mu_5 = \lambda/\mu_3 + \lambda/\mu_4$. This example is considered in Dai and Wang [11] for the FIFO scheduling policy. Here, as in the previous example, we restrict the scheduling policy

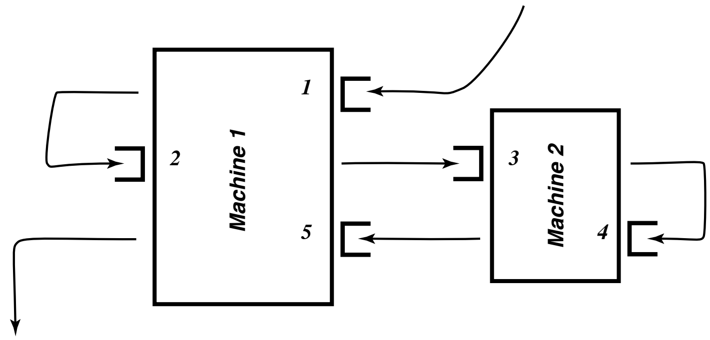

Figure 4. The Dai-Wang network

only to the class of non-idling policies. We wish to determine how large ρ may be while still guaranteeing system stability.

A piecewise linear function is again constructed using the algorithm presented in Theorem 4.2 and it is found that the LP has a solution indicating stability if $\lambda < 0.57190$ (approximately). In terms of the load factor, $\rho < (0.95)(0.57190) = 0.5433$. This same example is examined in [26], where it is found that for $\rho < 0.528$, a quadratic function may be found which guarantees stability.

It is of interest to determine if the piecewise linear approach completely specifies the stability region for this example. This question will be answered affirmatively in Section 6, but for illustrative purposes, the following special case is of interest. Consider the non-idling policy in which at machine 1, priority is given to the buffers in the order 5,1,2. At the second machine, priority is given to buffer 3. Furthermore, assume that the service times and interarrival times are deterministic. We performed a simulation of the system for various arrival rates, with the results for two particular values given in Figure 5. The plot on the left gives a typical sample path for a load slightly above the threshold $\rho = 0.5433$ determined by the piecewise linear LP. The other plot is a sample path in the case when the network is known to be stable since the load is slightly lower than this threshold.

5 Buffer priority policies

We now specialize the results of the previous section to that of buffer priority policies. As in Kumar and Meyn [26], suppose that at every machine σ there is a rank ordering of the set of buffers $\{x_i : \sigma(i) = \sigma\}$ served by the machine. To describe such a buffer priority policy more concisely, let $\{\theta(1), \dots, \theta(K)\}$ be a permutation of $\{1, 2, \dots, K\}$, with preference given to x_i over x_j if $\theta(i) < \theta(j)$ and both x_i and x_j share the same machine, i.e. $\sigma(i) = \sigma(j)$. The policy is non-idling and pre-emptive.

Consider the set of linear functions $\{V_j\}$ given by:

$$V_j(x) = \sum_{k=1}^{K} a_{j,k} x_k \qquad \text{for} \quad j = 1, 2, \dots, K,$$

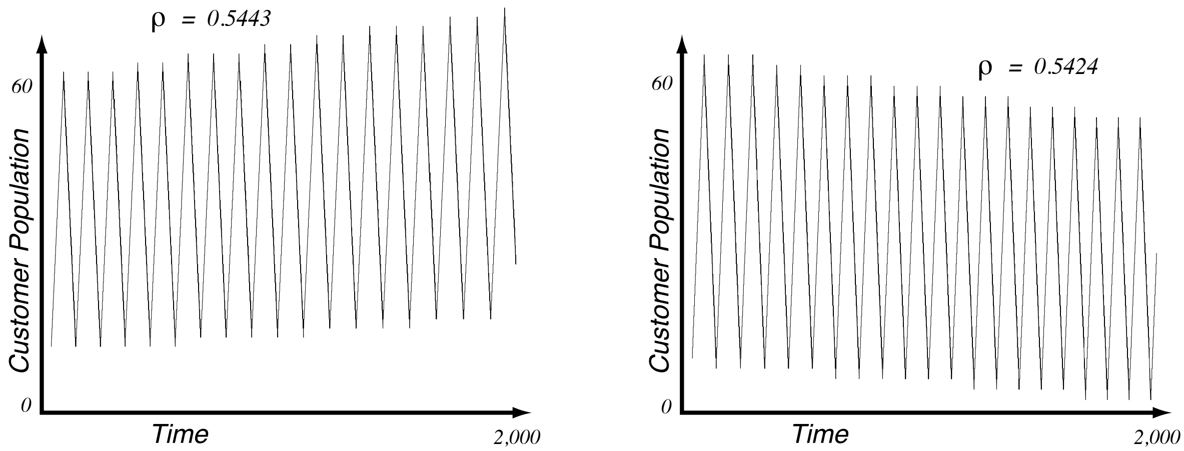

Figure 5. Simulation of the Dai-Wang network.

or equivalently, $V_j(x) = x^\top V^j$, where $V^j = (a_{j,1}, \ldots, a_{j,K})^\top$. To obtain a refinement of the drift and dominance conditions (22) and (23), define for $1 \le j \le K$,

$$\begin{aligned} P_j &:= \{i : \sigma(i) = \sigma(j) \text{ and } \theta(i) \le \theta(j)\} \\ \Upsilon_j &:= \{x \in \mathbb{R}_+^K : x_i = 0 \text{ for all } i \in P_j\}. \end{aligned} \tag{26}$$

The set P_j consists of indices of buffers which have priority greater than or equal to buffer j at machine $\sigma(j)$ and the face Υ_j is equal to the set of states for which all buffers which have priority greater than or equal to buffer j at $\sigma(j)$ are empty.

At buffer j we require that the function V_j exhibit a negative drift whenever any buffers of equal or higher priority at the same machine are nonempty:

$$\Delta(x)^\top V^j \le -1, \qquad x \in \Upsilon_j^c. \tag{27}$$

We also require the dominance condition, for all j,

$$V_j(x) < \max_{k \in P_j^c} V_k(x), \qquad x \in \Upsilon_j,\ x \ne 0. \tag{28}$$

That $V(x) = \max(V_j(x) : j = 1, \ldots, K)$ may be smoothed to form a valid Lyapunov function can be proved exactly as in Theorem 3.7. Analogous to Theorem 4.2, we may express the condition (27) and a sufficient condition for (28) as a linear program to obtain criteria for stability. A function V is *copositive* if $V(x) \ge 0$ for all x in the positive orthant.

Theorem 5.1 *Consider the following linear program:*

$$\max \gamma$$

subject to the constraints:

$$
\begin{array}{llcll}
\textit{(b1)} & \gamma & \leq & 1 & \\
\textit{(b2)} & \lambda a_{i,1} + r_i + \displaystyle\sum_{\{\sigma':\sigma'\neq\sigma(i)\}} s_{i,\sigma'} + \gamma \leq 0 & & & \textit{for all } i \\
\textit{(b3)} & r_i & \geq & \mu_j(a_{i,j+1} - a_{i,j}) & \textit{for all } i,\ j : \sigma(j) = \sigma(i),\ \theta(j) \leq \theta(i) \\
\textit{(b4)} & s_{i,\sigma'} & \geq & \mu_j(a_{i,j+1} - a_{i,j}) & \textit{for all } i,\ j : \sigma(j) = \sigma',\ \sigma(i) \neq \sigma' \\
\textit{(b5)} & a_{i,K+1} & = & 0 & \textit{for all } i \\
\textit{(b6)} & s_{i,\sigma'} & \geq & 0 & \textit{for all } i, \sigma' : \sigma(i) \neq \sigma' \\
\textit{(b7)} & \dfrac{1}{K - |P_i|} \displaystyle\sum_{k \in P_i^c} a_{k,l} & \geq & a_{j,l} + \gamma & \textit{for all } i,\ j : j \in P_i,\ l : l \in P_i^c
\end{array}
$$

If this LP has a solution $\gamma^* = 1$, then a piecewise linear function satisfying (27) and (28) exists.

(i) *If the resulting function V is copositive, then if Assumptions (A1)–(A3) hold, the network with buffer priorities is positive Harris recurrent, the queue length process $b_i(t)$ has a finite $(p-1)$-moment in steady state, and the limit (11) holds for each initial condition.*

If in addition (A4) holds, then the network model is V_*-uniformly ergodic so that (10) holds with $V_*(x) = \exp(\delta V(x))$ for some $\delta > 0$ and all $|f| \leq V_*$.

(ii) *If the resulting function V is not copositive, then if Assumptions (A1) and (A4) hold, the system is unstable in the sense that the system is highly non-robust in that the mean of the emptying time $\bar{\tau}$ is highly discontinuous in the initial condition,*

$$\sup_{\{(x,y):|x-y|\leq 2\}} |\mathsf{E}_x[\bar{\tau}] - \mathsf{E}_y[\bar{\tau}]| = \infty.$$

Proof The proof follows that of Theorem 4.2, in that (i) follows from Theorem 3.7 and Theorem 3.2. The result (ii) follows from Theorem 3.7 and Theorem 17 of Kumar and Meyn [28]. □

Note that the form of instability considered in Theorem 5.1 differs from that considered in Theorem 3.5. The stronger form of instability given in Theorem 3.5, the transience of the network, will be discussed in the next section.

The Dai-Wang network revisited We wish to test the effects of Lemma 4.1 in the more complex case in which the buffer priority policy is included as a constraint in the LP. If this sufficient condition is overly conservative then this may be apparent in high dimensional examples.

Consider the network of Figure 4. First, we consider the LBFS policy. Through the use of Theorem 5.1, it is found that the system is stable for $\rho_1, \rho_2 < 1$, which is exactly as expected from results developed concurrently in Dai and Weiss [12], and Kumar and Kumar [30]. This demonstrates that the additional constraints in the LP given in Theorem 5.1 result in a strictly larger stability region than obtained with the more general algorithm of Theorem 4.2.

Second, we consider the priority policy $\{5, 2, 1, 3, 4\}$, so that buffer five has highest priority at the first machine, while buffer three has priority at the second machine. With this policy the network is similar, though more complex than the Rybko-Stolyar

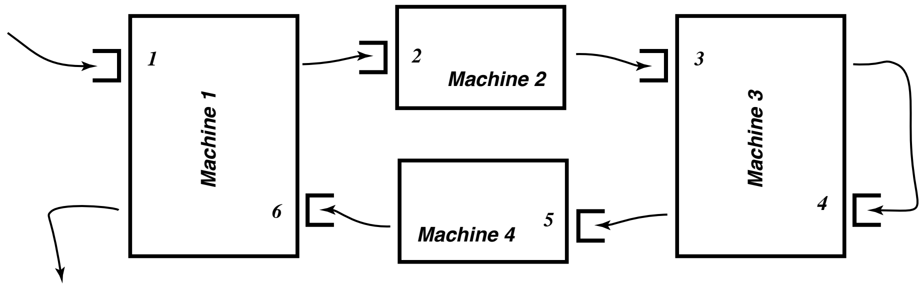

Figure 6. A re-entrant line with regulation.

example. Stability has been tested using Theorem 5.1, with numerical results yielding stability for $\rho_1 < 1$, $\rho_2 < 1$, and $\lambda/\mu_3 + \lambda/\mu_5 < 1$.

A regulator model Consider the network illustrated in Figure 6, consisting of four machines and six buffers. Let buffers 3 and 6 have priority at their respective machines. This model has been studied by Humes and Kumar [22, 43], where machines 2 and 4 are called *regulators*. A proper choice of service rates at the regulators will ensure stability of the system, even for a network which would otherwise be unstable. In the present example, letting μ_2, $\mu_5 \to \infty$ results in a system which is unstable whenever $\lambda/\mu_3 + \lambda/\mu_6 > 1$, even if $\rho_1, \rho_3 < 1$.

Using Theorem 5.1, we may determine the range of regulator service rates that stabilize the network. The particular parameters examined were $\lambda = 2$, $\mu_1 = \mu_4 = 10$ and $\mu_3 = \mu_6 = 3.5$. Note $\lambda/\mu_3 + \lambda/\mu_6 > 1$. If we let $\mu_2 = \mu_5 = \mu$, we would like to determine how large μ may be for the system to be stable (a lower bound is easily determined, $\mu > \lambda$). Numerically, it was determined that $\mu \leq 2.8242$ (approximately) results in stability, so the required range for μ is $2 < \mu \leq 2.8242$.

6 Instability of fluid limit models

In this section a specific test function approach is developed in order to apply Theorem 3.6. Two particular queueing networks will be studied as examples to demonstrate this method. To do this, associated fluid limit models will be examined and it will be found that if after some finite time, the fluid limit model evolves on an invariant, attracting subset of the state space, then appropriate test functions may be constructed to conclude instability for certain parameter values. The idea of examining reduced state space models is also used in Kumar and Kumar [45], and Malyšhev and Men'šikov [35]. The reduced state space will be denoted X_R. The precise structure of X_R will vary with the particular network topology.

For ease of exposition, it will be assumed that (A1) and (A4) hold. The results in this section may be generalized to the case when (A1), (A2), and (A3) hold. Let $Q(t) = x$, and consider the function $V(x) := \max_j V_j(x)$ with the constraints

$$\delta^{\mathsf{T}} V^j \quad \leq \quad -1, \qquad x \in \Upsilon_j^c \cap \mathsf{X}_R,\ \delta \in \delta(x); \tag{29}$$

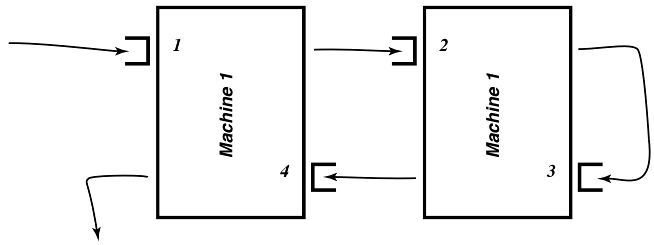

Figure 7. A simple network for instability analysis

$$V_j(x) \quad < \quad \max_{k \in P_j^c} V_k(x), \qquad x \in \Upsilon_j \cap \mathsf{X}_R,\ x \neq 0. \tag{30}$$

The following result relates (29) and (30) with stability and instability of the network model.

Theorem 6.1 *Consider the associated fluid limit model for a given queueing network with buffer priorities. Suppose a piecewise linear function V exists which satisfies (29) and (30). Suppose also that there exists $t_0 \geq 0$ such that $Q(t) \in \mathsf{X}_R$ for all $t \geq t_0$, for any fluid limit $\{Q, T\}$. Then*

(i) *If V is copositive, then*

$$\lim_{t\to\infty} t^n |\mathsf{E}_x[f(b(t))] - \mathsf{E}_\pi[f(b(0))]| = 0, \quad n \in \mathbb{Z}_+,$$

where f is any function on X which is bounded by a polynomial, i.e. for some constants $m \in \mathbb{Z}_+$, $c < \infty$, $|f(x)| \leq c(|x|^m + 1)$, $x \in \mathsf{X}$.

(ii) *If X_R does not contain the origin and if $-V$ is copositive, the network is unstable.*

Proof The result (i) follows from Theorem 3.4 above and Theorem 6.3 of Dai and Meyn [10] and similarly, (ii) follows from Theorem 3.6 and Theorem 3.5. □

We note that by using Theorem 6.1, one may construct a linear program to test for instability, just as in Theorem 5.1. The application of this result will now be demonstrated on two particular simple network models. Theorem 6.1 here has clear applications to more complex networks.

A two machine example Consider the network of Figure 7. At machine 1, buffer 4 has priority while at machine 2, buffer 2 has priority. The stability of this network and a variant have been extensively studied. However, it is worth considering as it is the simplest network for illustrating Theorem 6.1.

The instability of this network is investigated using an associated fluid limit model for the case when the capacity constraints are met, but the additional constraint $\lambda/\mu_2 + \lambda/\mu_4 < 1$ fails to hold. It is easy to check that this implies $\mu_1 > \mu_2$ and $\mu_3 > \mu_4$. The first step in the analysis is to calculate X_R. For the example under consideration, it is easily seen that for any initial condition, a valid solution to the integral equations (F1–F3) is that the state goes to zero and stays there. Clearly, if this solution were indeed a valid fluid limit, there would be no hope of showing instability using Theorem 6.1. This prompts the examination of the fluid limits themselves to find additional constraints. In particular, to avoid trajectories that tend to zero, the case when the fluid model is in the initial state $(x_1, 0, x_3, 0)$ must be examined, where $x_1, x_3 > 0$ and $x_1 + x_3 = 1$. This is due to the fact that this initial condition does not have a unique velocity vector and is reached from every initial condition. To see the second fact, let $V_1(x) = x_2 x_4$ and note that $\delta^\top(t) V_1'(Q(t)) \le -\mu_2 Q_4(t) - \mu_4 Q_2(t)$ for all t and any fluid limit with initial condition $Q_2(0), Q_4(0) > 0$. This implies that $V_1(Q(t)) = 0$ for some t and hence either buffer 2 or buffer 4 empties in finite time. Without loss of generality, assume buffer 2 is the one that empties. It is easy to see that due to the scheduling policy for the queueing network, buffer 2 remains empty until buffer 4 empties. That this property carries over to the fluid limit model is shown below.

For some $\varepsilon > 0$, it would be useful to show that either $T_2(t) = 0$ or $T_4(t) = 0$ for all $t \le \varepsilon$ if the initial state is of the form $(x_1, 0, x_3, 0)$, where $x_1, x_3 > 0$ and $x_1 + x_3 = 1$. For the queueing network model, let $Z_k(t)$ denote the number of times that buffer k empties in the interval $[0, t]$.

Lemma 6.2 *Consider the network of Figure 7 with buffer priorities $\{4, 1, 2, 3\}$. Let the sequence $\{x^n\}$ be such that $|x^n| \to \infty$, and assume*

$$\frac{x_1^n}{|x^n|} \to \alpha, \quad \frac{x_2^n}{|x^n|} \to 0, \quad \frac{x_3^n}{|x^n|} \to (1-\alpha), \quad \frac{x_4^n}{|x^n|} \to 0,$$

where $0 < \alpha < 1$. Then, with x^n equal to the initial condition of the network, there exists $\varepsilon > 0$ such that

$$\limsup_{n \to \infty} Z_k(|x^n| t) < \infty \quad a.s. \tag{31}$$

for $k = 2, 4$, and all $t \le \varepsilon$. Hence for any fluid limit with initial condition $(\alpha, 0, 1-\alpha, 0)$, either $T_2(t) = 0$ or $T_4(t) = 0$ on some small time interval containing the origin.

PROOF Without loss of generality, we focus our attention on buffer 2. Let Z_2^∞ (Z_4^∞) be the number of times that buffer 2 (buffer 4) empties if there is an infinite number of customers at buffer 1. For each n, for a fixed sample path, $Z_2(t) \le Z_2^\infty(t)$. Thus, we need only prove the result for Z_2^∞. Since $b_1(0) = \infty$, buffer 2 behaves like an unstable M/M/1 queue with a vacation taken each time that $b_2(t) = 0$ (corresponding to a busy period for buffer 4). It is obvious that an unstable M/M/1 queue empties only finitely often and this is equivalent to our claim that $Z_2^\infty < \infty$ a.s. □

Let

$$S_{ij} = \{x : x_i, x_j > 0, x_k = 0, k \ne i, j\},$$

and extend the definition of S_{ij} in a natural manner for a varying number of subscripts. With this and the results above, the following conclusions about the structure of the fluid limit model may be made. The reduced state space is given by the set of states where either buffer 2 or buffer 4 is empty, or more precisely,

$$\mathsf{X}_R = S_{123} \cup S_{23} \cup S_{12} \cup S_{134} \cup S_{14} \cup S_{13} \cup S_3 \cup S_1.$$

Also, the corresponding velocity vectors may be calculated. For example,

$$\delta((x_1, x_2, x_3, 0)) = (\lambda - \mu_1, \mu_1 - \mu_2, \mu_2, 0)$$

is single-valued, and $\delta((x_1, 0, x_3, 0))$ is multi-valued. In the latter case, Lemma 6.2 implies that the velocity vector can be defined by right continuity as either $(\lambda - \mu_1, \mu_1, 0, 0)$, or $(\lambda, 0, -\mu_3, \mu_3)$.

The instability properties of the network may now be examined using Theorem 6.1. It is found through the LP obtained by (29) and (30) that the network is unstable for $\lambda/\mu_2 + \lambda/\mu_4 > 1$.

The Dai-Wang example We now return to the Dai-Wang network introduced earlier. Essentially the same steps as in the previous example may be followed, with buffer 3 taking the role of buffer 2 in the last example, and buffer 5 taking the role of buffer 4. The reduced state space X_R may also be calculated in the same manner as the previous example, by noting that either buffer 3 or buffer 5 is empty in X_R. The corresponding velocity vectors may also be calculated. Using these observations one can construct an LP suggested by (29) and (30), which yields a function V with non-positive coefficients for $\lambda > .57190$ (approximately), and thus for these values of λ, the queueing network is unstable by Theorem 6.1. For this example, as in the previous one, the approach to classifying stability is tight.

7 Conclusion

We have presented an algorithm that allows the computer automation of the analysis of the stability and instability of re-entrant lines. Through various examples, it is suggested that this method completely characterizes the region of stability for the types of networks considered. An analysis of this conjecture would be important. If it proved that this method did not completely characterize stability, a modification of Lemma 4.1 would be worth examining, as it could very well improve the algorithm.

For the class of non-idling and buffer priority policies, piecewise linear functions of low complexity have been constructed. To construct such functions for other policies such as FIFO appears to be more difficult in general, due to the complex state representation of such models. However, an analysis of common policies that are not of the type examined in this paper would be a matter of interest.

The performance evaluation of queueing networks using quadratic test functions is considered in Kumar and Kumar [29], and Bertsimas et al. [7]. The question of whether piecewise linear functions can be applied to performance analysis is currently under investigation.

REFERENCES

[1] F. Baccelli and S. Foss. Ergodicity of Jackson-type queueing networks. *QUESTA*, 17:5–72, 1994.

[2] A. A. Borovkov. Limit theorems for queueing networks. *Theory Probab. Appl.*, 31:413–427, 1986.

[3] D.D. Botvich and A.A. Zamyatin. Ergodicity of conservative communication networks. Technical report, Institut National de Recherche en Informatique et en Automatique, Rocquencourt, 1993. Technical Report.

[4] M. Bramson. Instability of FIFO queueing networks. To appear in *Annals of Applied Probability*, 1995.

[5] H. Chen and A. Mandelbaum. Discrete flow networks: Bottlenecks analysis and fluid approximations. *Mathematics of Operations Research*, 16:408–446, 1991.

[6] D. Gamarnik D. Bertsimas and J. N. Tsitsiklis. Stability conditions for multiclass fluid queueing networks. Technical report, Massachusets Institute of Technology, 1995.

[7] D. Bertsimas, I. Ch. Paschalidis and J. N. Tsitsiklis. Scheduling of multiclass queueing networks: Bounds on achievable performance. In *Workshop on Hierarchical Control for Real–Time Scheduling of Manufacturing Systems*, Lincoln, New Hampshire, October 16–18, 1992.

[8] J. G. Dai. On the positive Harris recurrence for multiclass queueing networks: A unified approach via fluid limit models. *Ann. Appl. Probab.*, 5:49–77, 1995.

[9] J. G. Dai. The stability region of two-station queueing networks. Presented at the *Workshop On Stochastic Networks: Stability And Rare Events*, Center For Applied Probability, Columbia University, November 3 — November 4, 1995.

[10] J. G. Dai and S.P. Meyn. Stability and convergence of moments for multiclass queueing networks via fluid limit models. *IEEE Trans. Automat. Control*, 40:1889–1904, November 1995.

[11] J. G. Dai and Y. Wang. Nonexistence of Brownian models of certain multiclass queueing networks. *Queueing Systems: Theory and Applications*, 13:41–46, May 1993.

[12] J. G. Dai and G. Weiss. Stability and instability of fluid models for certain reentrant lines. *Math. Operations Res.*, 1995. to appear.

[13] D. Down and S. P. Meyn. The structure of fluid models and instability of queueing networks. presented at the *European Control Conference 1995*, Rome, Italy, 1995.

[14] D.G. Down and S. P. Meyn. Piecewise linear test functions for stability of queueing networks. In *Proceedings of the 33rd Conference on Decision and Control*, pages 2069–2074, Buena Vista, FL, December 1994.

[15] D.G. Down and S. P. Meyn. A survey of Markovian methods for stability of networks. In Guy Cohen and Jean-Pierre Quadrat, editors, *11th International Conference on Analysis and Optimization of Systems*, pages 490–504, Sophia-Antipolis, France, June 1994. Springer-Verlag.

[16] P. Dupuis and R. J. Williams. Lyapunov functions for semimartingale reflecting Brownian motions. *Ann. Appl. Probab.*, 22(2):680–702, 1994.

[17] G. Fayolle. On random walks arising in queueing systems: ergodicity and transience via quadratic forms as Lyapounov functions - part I. *Queueing Systems*, 5:167–183, 1989.

[18] G. Fayolle, V. A. Malyšhev, M.V. Men'šikov, and A. F. Sidorenko. Lyapunov functions for Jackson networks. Technical Report, INRIA Rocquencourt, 1991.

[19] S. Foss. Ergodicity of queueing networks. *Siberian Math. Journal*, 32:183–202, 1991.

[20] F. G. Foster. On the stochastic matrices associated with certain queueing processes. *Ann. Math. Statist.*, 24:355–360, 1953.

[21] J. M. Harrison and V. Nguyen. Brownian models of multiclass queueing networks: Current status and open problems. *QUESTA*, 13:5–40, 1993.

[22] C. Humes Jr. A regulator stabilization technique: Kumar & Seidman revisited. *IEEE Trans. Automat. Control*, AC-39:191–196, 1994.

[23] R. Z. Khas'minskii. *Stochastic Stability of Differential Equations.* Sijthoff & Noordhoff, Netherlands, 1980.

[24] J. F. C. Kingman. The ergodic behaviour of random walks. *Biometrika*, 48:391–396, 1961.

[25] P. R. Kumar. Re–entrant lines. *Queueing Systems: Theory and Applications*, 13:87–110, 1993.

[26] P. R. Kumar and S. P. Meyn. Stability of queueing networks and scheduling policies. *IEEE Transactions on Automatic Control*, 40(2):251–260, February 1995.

[27] P. R. Kumar and T. I. Seidman. Dynamic instabilities and stabilization methods in distributed real-time scheduling of manufacturing systems. *IEEE Trans. Automat. Control*, AC-35(3):289–298, March 1990.

[28] P.R. Kumar and S.P. Meyn. Duality and linear programs for stability and performance analysis queueing networks and scheduling policies. 1993. to appear, *IEEE Trans. Auto. Control*, November 1995.

[29] S. Kumar and P. R. Kumar. Performance bounds for queueing networks and scheduling policies. *IEEE Trans. Automat. Control*, AC-39:1600–1611, August 1994.

[30] S. Kumar and P. R. Kumar. Fluctuation smoothing policies are stable for stochastic re-entrant lines. To appear in *Discrete Event Dynamical Systems*, 1995.

[31] J. LaSalle and S. LefShetz. *Stability by Liapunov's Direct Method.* Academic, New York, NY, 1961.

[32] S. H. Lu and P. R. Kumar. Distributed scheduling based on due dates and buffer priorities. *IEEE Transactions on Automatic Control*, 36(12):1406–1416, December 1991.

[33] V. A. Malyšhev. Classification of two-dimensional positive random walks and almost linear semi-martingales. *Soviet Math. Dokl.*, 13:136–139, 1972.

[34] V. A. Malyšhev. Networks and dynamical systems. *Adv. Appl. Probab.*, 25:140–175, 1993.

[35] V. A. Malyšhev and M.V. Men'šikov. Ergodicity, continuity and analyticity of countable Markov chains. *Trans. Moscow Math. Soc.*, 1:1–48, 1982.

[36] M. V. Men'šikov. Ergodicity and transience conditions for random walks in the positive octant of space. *Soviet Math. Dokl.*, 15:1118–1121, 1974.

[37] S. P. Meyn. Transience of multiclass queueing networks via fluid limit models. *Ann. Appl. Probab.*, 1995. to appear.

[38] S. P. Meyn and D. Down. Stability of generalized Jackson networks. *Ann. Appl. Probab.*, 4:124–148, 1994.

[39] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability.* Springer-Verlag, London, 1993.

[40] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes II: Continuous time processes and sampled chains. *Adv. Appl. Probab.*, 25:487–517, 1993.

[41] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes III: Foster-Lyapunov criteria for continuous time processes. *Adv. Appl. Probab.*, 25:518–548, 1993.

[42] S. P. Meyn and R.L. Tweedie. A survey of Foster-Lyapunov techniques for general state space markov processes. In *Proceedings of the Workshop on Stochastic Stability and Stochastic Stabilization, Metz, France, June 1993*, 1994.

[43] J.R. Perkins, C. Humes Jr., and P.R. Kumar. Distributed control of flexible manufacturing systems: stability and performance. *IEEE Trans. on Robotics and Automation*, 37:132–141, 1994.

[44] A. N. Rybko and A. L. Stolyar. On the ergodicity of stohastic processes describing the operation of open queueing networks. *Problemy Peredachi Informatsii*, 28:3–26, 1992.

[45] S. Kumar and P. R. Kumar. Fluid limits: Cycles and instability. Technical report, Coordinated Science Laboratory, University of Illinois, Urbana, IL, 1995.

[46] T. I. Seidman. First come first serve can be unstable. *IEEE Transactions on Automatic Control*, 39(10):2166–2170, October 1994.

[47] K. Sigman. The stability of open queueing networks. *Stoch. Proc. Applns.*, 35:11–25, 1990.