

STABILITY OF MARKOVIAN PROCESSES II: CONTINUOUS-TIME PROCESSES AND SAMPLED CHAINS

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Abstract

In this paper we extend the results of Meyn and Tweedie (1992b) from discrete-time parameter to continuous-parameter Markovian processes Φ evolving on a topological space.

We consider a number of stability concepts for such processes in terms of the topology of the space, and prove connections between these and standard probabilistic recurrence concepts. We show that these structural results hold for a major class of processes (processes with continuous components) in a manner analogous to discrete-time results, and that complex operations research models such as storage models with state-dependent release rules, or diffusion models such as those with hypoelliptic generators, have this property. Also analogous to discrete time, 'petite sets', which are known to provide test sets for stability, are here also shown to provide conditions for continuous components to exist.

New ergodic theorems for processes with irreducible and countably reducible skeleton chains are derived, and we show that when these conditions do not hold, then the process may be decomposed into an uncountable orbit of skeleton chains.

IRREDUCIBLE MARKOV PROCESSES; ERGODICITY; RECURRENCE; RESOLVENTS; DIFFUSIONS

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1. Introduction

The purpose of this paper is to develop a structural and stability theory for continuous-time processes which is suitable for application in such areas as operations research, control and systems theory and communications theory; areas where complex stochastic models are frequently used, and where a Markovian state process can often be constructed.

Stability concepts, and the related ergodic theory, for continuous-time Markov processes have a large literature which includes many different approaches. The most easily applied and most complete theories have been developed for diffusion processes, or for processes with some form of regeneration points such as storage process with jumps from the origin. In the special case of diffusion processes, there

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has been considerable progress in classifying continuous-time processes through essentially sample path arguments. Underlying this analysis is the fact that the class of processes for which results have been obtained, those for which the generator of the process is *hypoelliptic*, possess strong Feller resolvent kernels (see [7], and also [6], [16], [17]); since the analysis of strong Feller chains in discrete time is relatively straightforward, and since the recurrence structure of the resolvent-chain and the recurrence structure of the process are essentially equivalent [3], [2], these results provide an immediate approach to the classification of the continuous-time process. The most general framework is in Kliemann [17], whilst Khas'minskii [16], [15] gives earlier treatments limited to strong Feller processes. The recent text by Kunita [19] also presents a brief but elegant development of strong Feller processes.

Strong Feller processes are, however, a relatively restricted class of processes. Our approach in the first part of this paper (Sections 2–4), where we develop structural results, is to identify properties following from the existence only of an everywhere non-trivial continuous component, as defined in [32], in which case we call Φ a T-process. Strong Feller processes are a subset of T-processes, but the latter allow a very much wider range of behavior, including jump processes as well as processes with diffusion-type paths, and processes which are perturbations on either type. This approach was also adopted for discrete-time chains in [24].

In Section 2, we formulate the key probabilistic forms of stability which we consider: these are continuous-time forms of Harris and positive Harris recurrence [26]. We then consider in Section 3 various forms of topological stability, analogous to those in [24], and show that for T-processes, there are detailed equivalences between the stochastic and the topological forms of stability.

These results follow from the identification of links between the recurrence structures for arbitrarily sampled chains and the process Φ , and these also enable us in Section 3.4 to investigate in detail both irreducible and reducible T-processes, and prove a new and stronger form of the Doeblin decomposition theorem. This extends results of [32].

Following these structural results, we then consider *petite sets*, a class of sets whose hitting times are known [22] to characterize the various forms of stability developed above; here we connect the T-process property and irreducibility with the existence of compact sets as petite sets. Hence for irreducible T-processes, compact sets may be used to classify a process as either recurrent or transient, positive recurrent or null recurrent. Using these petite set results, we extend the recent work in [22] to reducible or decomposable chains, and show that a Harris set exists for Φ whenever a Harris set exists for the chain obtained by sampling the process at renewal epochs which are independent of the process. This is considerably stronger than previous results in this direction in [32].

As examples of these chains, we show firstly that a diffusion process is a T-process if its generator is hypoelliptic, so that the decomposition theorem presented in [17] is a special instance of the results of the present paper; and secondly that

jump-deterministic processes, including general queueing, storage and risk processes are also irreducible T-processes under minimal conditions.

There are few results for general continuous-time processes which show that compact sets are ‘test sets’ for stability in this way, yet (as we see in [25]) such identification of test sets is crucial for classifying individual models. One approach which is similar to that used in this part of the paper may be found in the survey article [9]. Under condition (LSC) of [9], that the excessive functions for the resolvent-chain are lower semi-continuous, conditions for recurrence and transience are also obtained in terms of hitting probabilities to compact subsets of the state space. Condition (LSC) does, however, seem intrinsically harder to verify than the T-process condition, and in the special case of diffusion processes, the known conditions under which (LSC) is satisfied also imply that the resolvent for the process is strongly Feller.

The results of the sections above are concerned with characterizations of stability in various forms. It is perhaps more important to find out what if any useful properties may be found for appropriately stable processes, and it is this study which occupies the remainder of this paper.

We give in Section 6 verifiable sufficient conditions for ergodicity, and in Section 7 conditions under which we obtain convergence of the expectation $E[f(\Phi_t)]$ for *unbounded* functions f : these results are new and require different methods of proof, since the contraction properties of the total variation norm can no longer be employed as in [32], nor can the approach of assuming that the tail σ -field is trivial, developed in [27] and extended to continuous time in [8], be adopted.

Several of our new ergodic results depend heavily on the links between the process and its skeleton chains. In Section 5 we develop solidarity results for the skeleton chain, of interest in their own right and needed for our ergodic theorems. It is shown that if a skeleton chain possesses just one positive Harris set, then either the set is unique or an uncountable periodic orbit of Harris sets exists; and moreover, if the skeleton admits a countable recurrence structure, as it does for T-processes, then the periodic orbit is trivial.

Using this generalized aperiodicity result we extend our previous ergodicity results to non-irreducible processes. For the case of compact state space, we show that convergence takes place at a geometric rate.

In Part III of this series [25], we develop criteria for stability of continuous time processes based upon the infinitesimal generator and we also analyze the examples given here to demonstrate the value of the structural results in practice.

2. Probabilistic concepts

2.1. *Markov process concepts.* Here we present a minimal description of the models which we treat. Further details of the framework may be found in [5], [29]. We suppose that $\Phi = \{\Phi_t : t \in \mathbb{R}_+\}$ is a time-homogeneous Markov process with

state space $(X, \mathcal{B}(X))$, and transition semigroup (P^t) . The process Φ evolves on the probability space $(\Omega, \mathcal{F}, P_x)$, where $x \in X$ is the initial condition of the process, and Ω denotes the sample space. It is assumed that the state space X is a locally compact and separable metric space, and that $\mathcal{B}(X)$ is the Borel field on X . We assume that Φ is a Borel right process, so that in particular Φ is strongly Markovian with right-continuous sample paths [29]. The operator P^t acts on bounded measurable functions f and σ -finite measures μ on X via

$$P^t f(x) = \int P^t(x, dy) f(y), \quad \mu P^t(A) = \int \mu(dx) P^t(x, A)$$

so that the Markov property may be expressed

$$E_x[f(\Phi_{t+s}) | \mathcal{F}_s] = P^t f(\Phi_s), \quad s < t < \infty$$

where $\mathcal{F}_s = \sigma\{\Phi_u : 0 \leq u \leq s\}$.

Here the expectation operator E_x is the expectation corresponding to the probability measure P_x on sample space. When an event \mathcal{C} in sample space holds almost surely for every initial condition, we shall write ‘ \mathcal{C} holds a.s. $[P_*]$ ’.

For a measurable set A we let

$$\tau_A = \inf \{t \geq 0 : \Phi_t \in A\}, \quad \eta_A = \int_0^\infty \mathbf{1}\{\Phi_t \in A\} dt.$$

A Markov process is called φ -irreducible if for the σ -finite measure φ ,

$$\varphi\{B\} > 0 \Rightarrow E_x[\eta_B] > 0, \quad \forall x \in X.$$

The measure φ is then called an *irreducibility measure* for the process.

The set $A \in \mathcal{B}(X)$ is called *maximally absorbing* if $A \neq \emptyset$ and

$$x \in A \Leftrightarrow \{\eta_A = \infty\} \text{ a.s. } [P_*].$$

Maximally absorbing sets are utilized substantially in [32]. It follows from the Markov property that a maximally absorbing set A is *absorbing*, i.e. $P^t(x, A) = 1$, $x \in A$, $t \geq 0$. From this we know of no way in general to deduce that

$$(1) \quad P_x\{\Phi_t \in A \text{ for all } t \in \mathbb{R}_+\} = 1, \quad x \in A.$$

However, from Theorem 2.1 (ii) below, we do have $P_x\{\eta_{A^c} = 0\} = 1$ for any $x \in A$ when A is absorbing. Since Φ has right-continuous sample paths, it follows that (1) does hold if the set A is topologically closed. Consequently, any process can be restricted to a closed maximally absorbing set and remain a Borel right process.

2.2. Stochastic stability. Our first stability condition follows the standard definition of Harris recurrence, which is taken from [3].

Stochastic stability condition 1: Harris recurrence. The chain Φ is called Harris recurrent if either

- (a) for some σ -finite measure φ , $P_x\{\eta_A = \infty\} \equiv 1$ whenever $\varphi\{A\} > 0$; or
- (b) for some σ -finite measure μ , $P_x\{\tau_A < \infty\} \equiv 1$ whenever $\mu\{A\} > 0$.

The equivalence of the two definitions of Harris recurrence is given as Theorem 1.1 of [22]; see also [14] for another proof. The conditions of the second definition of Harris recurrence are much easier to verify than those of the first. We note however that simple counterexamples show that the measures μ and φ do not coincide in general (take for example the process (3) below). The proof relies on using an *exponentially sampled chain*: generalizations of this idea are central to this paper and are described in more detail in Section 2.3.

Clearly a Harris recurrent chain is φ -irreducible. For processes which are not φ -irreducible we need the related concept of a maximal Harris set: H is called *maximal Harris* if it is maximally absorbing, and for some measure φ on $\mathcal{B}(X)$,

$$\varphi\{A\} > 0 \Rightarrow P_x\{\eta_A = \infty\} = 1, \quad \text{for every } x \in H.$$

For Harris recurrent processes there is a second, and stronger, stochastic stability condition which we shall use. A σ -finite measure π on $\mathcal{B}(X)$ with the property

$$\pi\{A\} = \pi P^t\{A\} \triangleq \int \pi(dx) P^t(x, A), \quad A \in \mathcal{B}(X), \quad t \geq 0$$

will be called *invariant*. It is shown in [9] that if Φ is a Harris recurrent process then a unique (up to constant multiples) invariant measure π exists (see also [3]). If the invariant measure is finite, then it may be normalized to a probability measure, and in practice this is the main stable situation of interest.

Stochastic stability condition 2. Suppose that Φ is Harris recurrent with finite invariant measure π . Then Φ is called positive Harris recurrent.

One goal of this paper is to explore conditions for processes to be Harris recurrent and positive Harris recurrent, and to develop new or extended consequences of these forms of stability.

2.3. Structure of Φ and its sampled chains. In our analysis of Φ we will consider discrete-parameter Markovian processes (i.e. Markov chains) derived from Φ by a sampling process. The definitions of irreducibility, maximal Harris sets, and Harris recurrence have exact analogues for such discrete-parameter chains. See [23], [26], [28] for these concepts.

The central idea here is to consider the Markov process sampled at times $\{t_k : k \in \mathbb{Z}_+\}$, which form an undelayed renewal process which is independent of the Markov process Φ . The process $\{\Phi_{t_k}\}$ is a Markov chain evolving on X , whose recurrence properties under appropriate conditions (such as those in [32]) are closely related to those of the original process.

The simplest examples of this approach are through the Δ -skeleton and the resolvent kernel of Φ . The Δ -skeleton corresponds to the deterministic renewal sequence sampled at times $\Delta, 2\Delta, 3\Delta, \dots$. The transition kernel of this embedded chain is just $P^\Delta(x, A)$. The resolvent kernel $R : X \times \mathcal{B}(X) \rightarrow [0, 1]$ is defined as

$$R(x, A) = \int P^t(x, A) \exp(-t) dt.$$

For a fixed initial condition $x \in X$, it is then obvious that the transition function R is the law of a chain $\{\Phi_{t_k}\}$ where $\{t_k\}$ is a renewal process with an exponential increment distribution which is independent of Φ . We will call any discrete-time parameter chain with transition function R the R -chain, and let $\tilde{\tau}_C$ and $\tilde{L}(x, C) \triangleq P_x(\tilde{\tau}_C < \infty)$ denote respectively the first hitting time of C and the hitting probability for the R -chain. Operator theoretic definitions of \tilde{L} may be found in [26].

As a simple first example of the connection of the resolvent chain and the process we give a much weaker condition for the process to be irreducible.

Proposition 2.1. Suppose that there exists a σ -finite measure μ such that $\mu\{B\} > 0 \Rightarrow P_x[\tau_B < \infty] > 0$ for all x . Then Φ is φ -irreducible with $\varphi = \mu R$.

Proof. Let $\varphi(B) > 0$. Then by regularity of measures and Lusin's theorem there exists a compact set C and $\delta > 0$ such that $R(x, B) \geq \delta$ for $x \in C$ with $\mu(C) > 0$. But then for any x , we have by the strong Markov property, $E_x[\eta_B] \geq \inf_{y \in C} E_y[\eta_B] P_x(\tau_C < \infty) \geq \delta P_x(\tau_C < \infty)$.

Even closer connections between the resolvent chain and the process are exhibited in the following result, which we shall use several times and which is the basis of the equivalence of the two forms of Harris recurrence: part (ii) is proved in Theorem 2.3 of [22], and (i) is given in Brankovan [36].

Theorem 2.1. (i) For all $x \in X$ and $B \in \mathcal{B}(X)$,

$$\tilde{L}(x, B) = E_x[1 - \exp(-\eta_B)].$$

(ii) For all $B \in \mathcal{B}(X)$,

$$\lim_{t \rightarrow \infty} \tilde{L}(\Phi_t, B) = \lim_{t \rightarrow \infty} E_{\Phi_t}[1 - \exp(-\eta_B)] = \mathbf{1}\{\eta_B = \infty\} \quad \text{a.s. } [P_*].$$

Suppose now that a is a probability measure on \mathbb{R}_+ , and define the more general Markov transition function K_a as

$$(2) \quad K_a \triangleq \int P^t a(dt).$$

If a is the increment distribution of the undelayed renewal process $\{t_k\}$, then K_a is again the transition function for the Markov chain $\{\Phi_{t_k}\}$.

For an arbitrary Markov transition function K , we call the associated Markov chain the K -chain. When a is a general exponential distribution with mean α^{-1} we will write R_α for K_a . In [32] it is shown in some generality that the recurrence structure of Φ and the K_a -chain are identical. In particular, under suitable conditions on a , their Harris sets coincide.

Such connections do not hold for all sampled chains. A virtually canonical example of the problems encountered in connecting sampled chains and the original process is given by the so-called 'clock process'. Consider the deterministic uniform

motion on the unit circle S^1 in the complex plane described by the equation

$$(3) \quad \Phi_t = \exp(2\pi it)\Phi_0, \quad t \geq 0.$$

Then for the probability a which assigns unit mass at $\{1\}$, each individual state $x \in S^1$ is a Harris set for the K_a -chain, so the K_a -chain possesses an uncountable number of Harris sets, but the process is a μ^{Leb} -irreducible Harris process.

The results of Theorem 2.1 do not hold without some restrictions on either the probability a , or the continuity properties (in the time parameter) of the transition kernels P^t of the process, such as those imposed in [32]. Below, however, we will weaken these conditions considerably.

Many of the results for sampled chains depend on links between resolvents and sampled chains with a possessing a bounded density. The next result illustrates some of these connections, and those with the structure of Φ itself.

Proposition 2.2. (i) Suppose that the probability a on $(0, \infty)$ possesses a bounded density with respect to Lebesgue measure. Then there exists a constant $0 < M < \infty$ such that

$$K_a(x, B) \leq M\check{L}(x, B), \quad \forall x \in X, \quad B \in \mathcal{B}(X).$$

(ii) Φ is φ -irreducible if and only if the R -chain is φ -irreducible.

(iii) If $B \in \mathcal{B}(X)$ is absorbing (i.e. B is non-empty and $P^t(x, B) = 1$ for $x \in B$, $t \in \mathbb{R}_+$) then the set $\{x \in X : \check{L}(x, B) = 1\}$ is maximally absorbing.

(iv) If H is a Harris set for the R -chain, then $\{x \in X : \check{L}(x, H) = 1\}$ is a maximal Harris set for the process Φ .

Proof. (i) is shown as Proposition 3.1 of [22] and (ii) follows directly from Theorem 2.1 (i) and the definitions. To prove (iii), let $B_1 = \{x \in X : \check{L}(x, B) = 1\}$. Since the set B is absorbing for the R -chain, $B_1 = \{x \in X : \check{L}(x, B_1) = 1\}$. It follows from Theorem 2.1 (ii) that

$$B_1 = \{x \in X : P_x\{\eta_{B_1} = \infty\} = 1\},$$

which is the desired result. Result (iv) now follows from (iii) and Theorem 2.1 of [32].

For the discrete-time R -chain, an irreducibility measure ψ is called *maximal* if $\varphi < \psi$ (φ is absolutely continuous with respect to ψ) for any other irreducibility measure φ . From Proposition 2.2 (ii) such a maximal measure also exists for continuous-time processes, and we reserve the symbol ψ exclusively for maximal irreducibility measures. The collection of sets $A \in \mathcal{B}(X)$ for which $\psi(A) > 0$ will be denoted $\mathcal{B}^+(X)$, and a set $A \in \mathcal{B}(X)$ for which $\psi(A^c) = 0$ will be called *full*.

In practice, the exact duplication of recurrence structures between Φ and its sampled chains is less important than identifying the existence of Harris sets for Φ from those of a sampled chain such as R as in Proposition 2.2.

One of our main aims is to prove that a Harris set exists for the process if a Harris set exists for the K_a -chain for any probability measure a , without any additional restrictions on the transition probabilities of Φ , even if (as with the clock process) it does not coincide with the Harris set for the K_a -chain. We state the next result to indicate the links we will expect between sampled chains and the recurrence structure of the continuous-time processes, and to motivate their development.

Theorem 2.2. Suppose that a is a general probability measure on \mathbb{R}_+ . Then

- (i) if the K_a -chain is Harris recurrent, then so is the process Φ ;
- (ii) if a Harris set H_0 exists for the K_a -chain, then a maximal Harris set H exists for Φ , with $H_0 \subset H$, where the inclusion may be strict;
- (iii) suppose that a has a finite mean, that H_0 is a Harris set for the K_a -chain, and that the invariant measure π_0 which is supported on H_0 is finite. Then the maximal Harris set H containing H_0 also supports a finite invariant measure π which may be expressed as

$$\pi\{B\} = \int_0^\infty E_{\pi_0}[\mathbf{1}\{\Phi_t \in B\}a[t, \infty)] dt, \quad B \in \mathcal{B}(X).$$

Proof. The first statement is proved as Theorem 3.1 of [22], and extends Theorem 2.2 of [32] which required additional conditions on the process or on a due to the use of the first form of Harris recurrence; it is however a straightforward consequence of the second form of Harris recurrence.

The remainder of the theorem is new: it is a consequence of the more general Theorem 4.3 presented in Section 4.3, and so we postpone the proof until then.

Sampled chains provide these links, and also provide the crucial tool in equating recurrence concepts with topological stability concepts, as we now show.

3. Topological stability and continuous components

3.1. *Topological stability concepts.* The following stability concepts are continuous-time versions of those used in Meyn and Tweedie [24]. As in that paper, we say that a trajectory *converges to infinity*, which shall be denoted $\{\Phi \rightarrow \infty\}$, if $\Phi_t \in C^c$ for any compact set $C \subset X$, and all $t \in \mathbb{R}_+$ sufficiently large. Due to our assumptions on the state space, the event $\{\Phi \rightarrow \infty\}$ is a measurable set: if $\{O_n : n \in \mathbb{Z}_+\}$ denotes a sequence of open precompact sets whose union is equal to X , then by right continuity the event $\{\Phi \rightarrow \infty\}$ may be expressed as

$$\{\Phi \rightarrow \infty\} = \bigcap_{n=1}^\infty \left\{ \bigcup_{m=0}^\infty \{\theta^m \eta_{O_n} = 0\} \right\}.$$

Topological stability condition 1: non-evanescence. A Markov chain Φ is called non-evanescent if $P_x\{\Phi \rightarrow \infty\} = 0$ for each $x \in X$.

This condition is introduced for discrete-time chains in Meyn and Tweedie [24]. Non-evanescent processes are called non-explosive by Meyn [20], but this nomenclature is at odds with the notion of explosion in a finite time for continuous-time processes; and non-dissipative by Tweedie [33], although regrettably, the analogous solutions to stochastic differential equations are called dissipative by Khas'minskii [16].

We also consider a stronger concept, boundedness in probability, introduced in [16], and the related concept, *boundedness in probability on average*, used in [21].

Topological stability condition 2: boundedness in probability. The process Φ is called bounded in probability if for each initial condition $x \in X$ and each $\epsilon > 0$, there exists a compact subset $C \subseteq X$ such that

$$\liminf_{t \rightarrow \infty} P_x\{\Phi_t \in C\} \geq 1 - \epsilon.$$

The process Φ is called bounded in probability on average if for each initial condition $x \in X$ and each $\epsilon > 0$, there exists a compact subset $C \subseteq X$ such that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t P_x\{\Phi_s \in C\} ds \geq 1 - \epsilon.$$

The condition of boundedness in probability will be seen to imply the existence of an invariant probability under suitable continuity conditions on the transition kernels P^t . One such is the *weak Feller* property: Φ is called weak Feller if the function $P^t f$ is continuous for each $t > 0$ whenever f is bounded and continuous. The first and most elementary consequence of boundedness in probability is the following.

Theorem 3.1. If Φ has the weak Feller property and is bounded in probability on average, then an invariant probability measure exists for Φ .

Proof. This is essentially a consequence of the main result of [4].

Consequently, if Φ is a Harris-recurrent process which is bounded in probability on average, then it is positive Harris recurrent provided it is also weak Feller. We now go on to consider a large class of processes for which boundedness in probability on average and positive Harris recurrence are in fact equivalent.

3.2. Continuous components and T-processes. A kernel T is called a *continuous component* of a function $K : (X, \mathcal{B}(X)) \rightarrow \mathbb{R}_+$ if

- (i) For $A \in \mathcal{B}(X)$ the function $T(\cdot, A)$ is lower semi-continuous;
- (ii) For all $x \in X$ and $A \in \mathcal{B}(X)$, the measure $T(x, \cdot)$ satisfies $K(x, A) \geq T(x, A)$.

The continuous component T is called *non-trivial* at x if $T(x, X) > 0$. This definition of a continuous component is taken from [32]. We will be concerned primarily with continuous components of the Markov transition function K_a , as defined in (2).

T-processes. A process will be called a T-process if for some probability a , the K_a -chain admits a continuous component T which is non-trivial for all $x \in X$.

Several examples of T-processes are given below, including diffusion processes in Section 3.3 and jump-deterministic models in Section 4.2.

Continuous components of the hitting probability for the R -chain will play an important role in developing the connections between the stability of the process and that of the R -chain. The following results closely link components of Φ with those of the R -chain.

Proposition 3.1. (i) If Φ is a T-process, then there exists a probability a on $(0, \infty)$ which possesses a bounded density with respect to Lebesgue measure, and for which K_a possesses an everywhere non-trivial continuous component.

(ii) If Φ is a T-process then \check{L} possesses an everywhere non-trivial continuous component.

Proof. To prove (i), suppose that Φ is a T-process. Hence there exists a probability b on \mathbb{R}_+ for which K_b possesses an everywhere non-trivial continuous component S . Then letting $a = b * e_1$, where e_1 is the standard exponential distribution, and $T = SR$, it is easy to see that T is a continuous component of K_a , that $T(x, X) = S(x, X)$ for all $x \in X$, and that a possesses a smooth, bounded density.

Result (ii) follows directly from (i), and Proposition 2.2 (i).

Using this we can show that the R -chain and the process evanesce with identical probabilities.

Proposition 3.2. Let Φ be a T-process. For any fixed initial condition $x \in X$

$$\{\check{\Phi}_k \rightarrow \infty \text{ as } k \rightarrow \infty\} = \{\Phi_t \rightarrow \infty \text{ as } t \rightarrow \infty\} \quad \text{a.s. } [P_x].$$

Proof. From Theorem 2.1 (i) we have

$$(4) \quad \check{L}(x, B) = E_x[1 - \exp(-\eta_B)].$$

It follows from Proposition 3.1 of [28] that for any measurable set B ,

$$(5) \quad \lim_{n \rightarrow \infty} \check{L}(\check{\Phi}_n, B) = \mathbf{1}\{\check{\Phi}_n \in B \text{ i.o.}\} \quad \text{a.s.}$$

Suppose that the sample path $\{\check{\Phi}_k\}$ converges to infinity. Let $C \subset X$ be a compact set, and choose B compact so that for some $\varepsilon > 0$ and any $x \in C$,

$$(6) \quad \check{L}(x, B) \geq \varepsilon$$

This is possible by Proposition 3.1, which asserts that \check{L} possesses an everywhere non-trivial continuous component.

Suppose that $\theta' \eta_C > 0$ for some sample path and all $t \in \mathbb{R}_+$. Then it follows from (6) and Theorem 2.1 (ii) that along this sample path

$$\lim_{t \rightarrow \infty} \check{L}(\Phi_t, B) = 1,$$

and so by (5), $\Phi_{t_n} \in B$ for infinitely many $n \geq 1$, contrary to our assumption. Hence for any compact set C , $\theta^t \eta_C = 0$ for all t sufficiently large, or in the notation introduced above, $\Phi \rightarrow \infty$.

In the proof we use very weak properties of T-processes, and the result is valid if the component T is only weakly continuous [31]. Hence Proposition 3.2 is also valid for Feller processes.

We now have the following equivalences.

Theorem 3.2. Suppose that Φ is a φ -irreducible T-process. Then

- (i) Φ is Harris recurrent if and only if Φ is non-evanescent;
- (ii) Φ is positive Harris recurrent if and only if Φ is bounded in probability on average.

Proof. If Φ is Harris recurrent, then it trivially follows that Φ is non-evanescent. Conversely, if Φ is non-evanescent, then by Proposition 3.2 the R -chain is also non-evanescent. By Proposition 2.2 the R -chain is irreducible, and \check{L} possesses an everywhere non-trivial continuous component. Hence Harris recurrence of the R -chain follows from the remark following the Doeblin decomposition theorem of Meyn and Tweedie [24]. Harris recurrence of the process then follows from Harris recurrence of the R -chain using Theorem 2.1.

If Φ is positive Harris recurrent then, by the ergodic theorem for Harris recurrent Markov processes (cf. [3], p. 169), the process is bounded in probability on average. Conversely, suppose Φ is bounded in probability on average. Then it is non-evanescent, and by (i) it follows that the process is Harris recurrent, and we now show positivity from the ergodic theorem again. For if Φ is a T-process which is bounded in probability on average then by the same method of proof that is used in Theorem 4.1 (iii) of Meyn and Tweedie [24], any compact set has finite π -measure. Suppose by way of contradiction that the invariant measure π is not finite: by the ergodic theorem of [3] we would then have the limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t P^s(x, C) = 0$$

for any compact set C . Hence π is finite, giving positivity as required.

These equivalences extend to non-irreducible processes: Theorem 8.2 which will be given in Section 8 has the details.

3.3. *Diffusions which are T-processes.* In this section we show that a large and frequently studied class of diffusions provides examples of T-processes if the noise can drive the process to a sufficiently large set of states

Consider the diffusion process (cf. [13])

$$(7) \quad d\Phi_t = Y(\Phi_t) dt + \sum_{i=1}^m X_i(\Phi_t) \circ dB_t$$

where \circ denotes the Stratonovich stochastic integral. We assume Φ_0 is independent of (B_t) ; X is a C^∞ , σ -compact, connected and orientable n -dimensional manifold; and the vector fields (Y, X_1, \dots, X_m) are C^∞ . The process Φ is a weak Feller, strong Markov process with continuous sample paths (and hence also a right process) with generator

$$(8) \quad \mathcal{A} = Y + \frac{1}{2} \sum_{i=1}^m X_i^2,$$

as shown in [13]. Let \mathcal{L} denote the Lie algebra generated by $\{Y, X_1, \dots, X_m\}$, and \mathcal{L}_0 the ideal in \mathcal{L} generated by $\{X_1, \dots, X_m\}$. The operator \mathcal{A} is called *hypoelliptic* if

$$(H) \quad \dim \mathcal{L}(x) = n, \quad x \in X$$

and we say that \mathcal{A} satisfies Condition (E) if

$$(E) \quad \dim \mathcal{L}_0(x) = n, \quad x \in X$$

In many instances Conditions (E) and (H) are equivalent [13]. They are related to the T-process property by the following result.

Theorem 3.3. For the diffusion Φ defined in (7):

(i) If (H) holds then the resolvent has the strong Feller property, and is thus its own continuous component;

(ii) If (E) holds then the Markov transition function P^Δ has the strong Feller property for each $\Delta > 0$, and is hence its own continuous component.

Thus under (H) or (E), Φ is a T-process.

Proof. In the proof of Lemma 3.1 of [17], it is shown that when \mathcal{A} is hypoelliptic, a result of [7] implies that the resolvent operator R has the strong Feller property.

This gives (i); the result (ii) follows from Theorem 3 of [13].

Although the strong Feller property is found for hypoelliptic diffusions in [17], it is not utilized in the subsequent results of that paper, and our results appear to give a powerful and essentially novel approach to the analysis of such diffusions.

One particular example of such diffusions, whose asymptotic properties we will study in Meyn and Tweedie [24], is the class of linear systems under memoryless non-linear control. For these there are more specific conditions ensuring that the model is a T-process.

Let $X = \mathbb{R}^n$, and define Φ as the state process for a linear system under memoryless non-linear control, following [35], by

$$(9) \quad d\Phi_t = F\Phi_t dt - b\phi(c^T\Phi_t) dt + G(\Phi_t) dw_t.$$

Proposition 3.3. Suppose that for a linear system under memoryless non-linear control, the functions G and ϕ are C^∞ , and there exists a constant B for which

$$(10) \quad 0 < G(x)G(x)^T \preceq BI, \quad x \in X.$$

Then the model is an irreducible T-process.

Proof. From [18] we know that for the vector fields defined in (7) the following identities hold:

$$\begin{aligned} \mathcal{L}_0 &= \mathcal{L}(ad^r(Y)X_j; j = 1, \dots, m, r = 0, 1, 2, \dots) \\ &= \left\{ \sum_{i=1}^m \lambda_i X_i + Z : \lambda_i \in \mathbb{R}, Z \in [\mathcal{L}, \mathcal{L}] \right\}, \end{aligned}$$

where $ad(Y)X = [Y, X]$, and $[\mathcal{L}, \mathcal{L}]$ is the derived algebra of \mathcal{L} . Using these identities and Theorem 3.3 one can see that in fact Φ has the strong Feller property, so that the T-process condition is immediately satisfied.

Irreducibility also follows from the conditions of the proposition, which imply that the support of Φ_t is equal to \mathbb{R}^n for all $t > 0$ (see the remarks following Proposition 5.2 of [18]).

3.4. The Doeblin decomposition. In [32] it is shown that the state space of a T-process may be broken into a countable union of disjoint Harris sets, together with a σ -transient set. Here we strengthen these results. As in the discrete parameter case, a sample path of Φ either evanesces, or enters some maximal Harris set with probability 1.

Theorem 3.4 (decomposition theorem). Suppose that Φ is a T-process. Then

(i) The state space may be decomposed into the disjoint union

$$X = \sum_{i \in I} H_i + E$$

where the index set I is countable, and each H_i is a maximal Harris set with associated invariant measure π_i ;

(ii) For each compact set $C \subset X$, $H_i \cap C = \emptyset$ for all but a finite number of $i \in I$;

(iii) For each initial condition $x \in X$,

$$P_x \{ \{ \Phi \rightarrow \infty \} \cup \{ \eta_A = \infty \} \} = 1$$

whenever the set A satisfies $\pi_i\{A\} > 0$ for each $i \in I$.

(iv) If Φ is non-evanescent then $P_x \{ \eta_H = \infty \} \equiv 1$, where $H = \sum H_i$. For each $i \in I$, the Harris set H_i is positive if and only if

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t P^s(x, C) ds > 0$$

for some $x \in H_i$, and some compact set $C \subset X$.

Proof. Results (i) and (ii) follow from the decomposition theorems presented in [32], [24]. To prove (iii), we apply Proposition 3.1 and the decomposition theorem of Meyn and Tweedie [24]. Note that the remark on p. 550 of that paper allows us to conclude that since \tilde{L} possesses an everywhere non-trivial continuous component, every sample path of the R -chain either converges to ∞ , or enters and remains in a maximal Harris set.

Let $C \subset \sum H_i$ be a closed set for which $\pi_i\{C\} > 0$ for all $i \in I$. Then for the R -chain, every trajectory either converges to infinity, or enters the set C . By Proposition 3.2 we conclude that $P_x\{\{\Phi \text{ enters } C\} \cup \{\Phi \rightarrow \infty\}\} = 1$. Since $\Phi_{\tau_C} \in \sum H_i$, and hence $P_{\Phi_{\tau_C}}\{\eta_A = \infty\} = 1$, the proof is completed by conditioning on \mathcal{F}_{τ_C} , and using the strong Markov property. Result (iv) is proved using the same argument used in the proof of Theorem 3.2 (a).

From the decomposition theorem we see that if Φ is a T-process, then Φ is non-evanescent if and only if every trajectory enters some Harris set with probability 1. This result extends Theorem 3.1 of [32] for general processes.

In the special case of hypoelliptic diffusions, we know from Section 3.3 that Φ is a T-process. The ergodic decomposition of [17] is then obviously a special case of the general T-process result, although the methods of proof in [17] are different. Moreover, because we only require T-processes to have a continuous *component*, it follows that the results for the processes with hypoelliptic generators flow through to *permutations* of such processes, allowing for analysis of much more general processes such as those with superimposed jumps.

We now go on to explore the connections between T-processes and petite sets, which were shown in Meyn and Tweedie [24] to be crucial in discrete time.

4. Petite sets and continuous components

4.1. *Petite sets and the existence of continuous components.* The class of *petite sets* will be seen to play the same role as the *small sets* of [26]. In particular, below and in Meyn and Tweedie [25] we show that they are test sets or ‘status sets’ [35] for Harris recurrence, as they are in discrete time (see Meyn and Tweedie [24]).

Adapting the definition used in Meyn and Tweedie [24] we say that a non-empty set $C \in \mathcal{B}(X)$ is ν_a -*petite* if ν_a is a non-trivial measure on $B(X)$, a is a probability measure on $(0, \infty)$, and $K_a(x, \cdot) \cong \nu_a(\cdot)$ for all $x \in C$. The set C will be called simply *petite* when the specific measure ν_a is unimportant. Well-behaved petite sets exist in quantities when the process is φ -irreducible. From Proposition 3.2 of [22] we have the following result.

Proposition 4.1. If Φ is φ -irreducible then

(i) the state space may be expressed as the union of a countable collection of petite sets;

(ii) if C is ν_a -petite then for any $\alpha > 0$ there exists an integer $m \geq 1$ and some maximal irreducibility measure ψ_m such that the set C is ψ_m -petite for the R_α^m -chain.

As in the discrete-parameter case, there exists a close connection between the existence of petite sets possessing desirable topological properties and continuous components. The following result is the continuous version of Propositions 5.1 and 5.2 in Meyn and Tweedie [24] and may be proved in exactly the same way as in the discrete-parameter case.

Proposition 4.2. (i) If an open petite set C exists, then K_a possesses a continuous component for some a , non-trivial on all of C .

(ii) Suppose that for each $x \in X$ there exists a probability a_x on \mathbb{Z}_+ such that K_{a_x} possesses a continuous component T_x which is non-trivial at x . Then Φ is a T-process.

For processes which are non-evanescent, there is a particularly strong connection between the existence of continuous components and petite sets: we have the following result.

Theorem 4.1. (i) Suppose that $P_x\{\Phi \rightarrow \infty\} < 1$ for one x . Then every compact set is petite if and only if Φ is an irreducible T-process.

(ii) Suppose that $P_x\{\Phi \rightarrow \infty\} < 1$ for all $x \in X$, and that Φ is a T-process. Then every compact set admits a finite cover by open petite sets.

(iii) If $P_x\{\Phi \rightarrow \infty\} < 1$ for all $x \in X$, and if Φ is a T-process, then in particular the resolvent R-chain is a T-chain.

Proof. Parts (i) and (ii) are exactly as in Theorem 5.1 of Meyn and Tweedie [24]. To prove (iii), we first apply Proposition 4.1 to obtain for each $i \in I$ a petite set C_i for the R-chain with $\pi_i(C_i) > 0$. By the Decomposition Theorem 3.4 we have $R(x, \cup C_i) > 0$ for every $x \in X$, and hence, because Φ is a T-process, there exists a probability a such that

$$K_a R(x, \cup C_i) \geq TR(x, \cup C_i) > 0$$

for all $x \in X$. Hence the open sets O_{ij} defined by

$$O_{ij} \triangleq \{x \in X : TR(x, C_j) \geq i^{-1}\}, \quad i \geq 1, j \in I$$

form an open cover of X . We now show that each of these sets is petite for the R-chain. This property together with Theorem 5.1 of Meyn and Tweedie [24] will complete the proof. To see this, observe that for any $x \in O_{ij}$,

$$\int_0^\infty P^t R(x, C_j) a(dt) \geq i^{-1}$$

and hence for some N sufficiently large and all $x \in O_{ij}$,

$$\int_0^N P^t R(x, C_j) a(dt) \geq (2i)^{-1}.$$

Since $P^t R \leq e^t R$ this shows that for each $x \in O_{ij}$, $R(x, C_j) \geq [2ie^N]^{-1}$. Since C_j is petite for the R-chain, this shows that O_{ij} is also petite for the R-chain.

4.2. Petite sets and a jump-deterministic T-process. In practice there are two ways of determining whether a process is a T-process. One can check the Feller properties of the process itself, or (often more easily) the resolvent of the process: if the strong Feller property holds, or some modification of it, then there is a direct construction

of the continuous component. This method was used for the diffusions in Section 3.3.

The other standard way in which a continuous component occurs is by the existence of a regenerative state $\{x^*\}$ such that the time to reach $\{x^*\}$ is bounded below in a continuous way. It then follows as in Lemma 3.1 of Meyn and Tweedie [24] that the sets for which the time to reach $\{x^*\}$ is bounded from zero are petite, and hence if $\{x^*\}$ is reachable from every initial condition then we have a T-process from Proposition 4.2. We demonstrate this approach with a relatively complex storage model.

In [11], [12] Harrison and Resnick consider storage processes with compound Poisson input and a general deterministic release path between jumps of the compound Poisson process. These are defined by the storage equation

$$(11) \quad \Phi_t = x + A(t) - \int_0^t r(\Phi_s) ds, \quad t \geq 0$$

where $\{A(t), t \geq 0\}$ is a compound Poisson process with rate λ and jump size distribution $H(\cdot)$, not degenerate at zero, and we assume the release function $r(\cdot)$ is strictly positive, left continuous and has a positive right limit everywhere in $(0, \infty)$, with $r(0) = 0$.

We show here that in general such jump-deterministic models are φ -irreducible T-processes, and in Meyn and Tweedie [25] give conditions for them to be recurrent, ergodic and geometrically ergodic. As in [11], we assume further that for one and hence all $a > 0$

$$(12) \quad 0 < R(a) \triangleq \int_0^a [r(u)]^{-1} du < \infty.$$

Theorem 4.2. For any a with $a(0) < 1$, the K_a -chain is an irreducible T-process when (12) holds.

Proof. Let $T_t(x, 0) = P_x\{\{\Phi_t = 0\} \cap \{A(t) = 0\}\}$; that is, $T_t(x, 0)$ is the probability the system is empty at time t , and no jumps have occurred in $[0, t]$. Then

$$T_t(x, 0) = \begin{cases} \exp(-\lambda t), & t \geq R(x) \\ 0, & t < R(x) \end{cases},$$

and so if we define, for $a^* = \sum \alpha_n a^{n*}$, with $\alpha_n > 0$ for all n ,

$$T(x, \{0\}) = \int_0^\infty T_t(x, 0) a^*(dt), \quad T(x, X \setminus \{0\}) = 0$$

then $T(x, X) > 0$ for all x since a^* has unbounded support. Moreover

$$K_{a^*}(x, 0) \geq \int_0^\infty T_t(x, 0) a^*(dt) = \int_{R(x)}^\infty \exp(-\lambda t) a^*(dt)$$

and since $R(x)$ is increasing and continuous, T is a continuous component of K_a . By definition of T-processes in discrete time, the K_a -chain is thus a T-process. The irreducibility is trivial with δ_0 as the irreducibility measure when (12) holds.

4.3. *Petite sets and Harris recurrence.* The purpose of this section is to develop characterizations of Harris recurrence in terms of the finiteness of the hitting time to a single petite set.

Theorem 4.3. Suppose that C is petite.

(i) If

$$P_{x_0} \left\{ \limsup_{t \rightarrow \infty} \mathbf{1}_C(\Phi_t) = 1 \right\} = 1$$

for some $x_0 \in X$, then a Harris set exists which contains the state x_0 .

(ii) If $P_x \{ \tau_C < \infty \} = 1$ for all $x \in X$ then Φ is Harris recurrent.

Proof. (i) We have that C is φ_a -petite. Let b denote a probability on \mathbb{R}_+ with a bounded, continuous density, and define $\varphi = \varphi_a K_b$, so that

$$K_{a*b}(x, B) = K_a K_b(x, B) \cong \varphi_a K_b \{ B \} = \varphi \{ B \}, \quad x \in C.$$

We will demonstrate that

$$(13) \quad \varphi \{ B \} > 0 \Rightarrow \eta_B = \infty \quad \text{a.s.}$$

when the initial condition $\Phi_0 = x_0$.

By Proposition 2.2 (i), there exists $M < \infty$ for which

$$K_{a*b}(x, B) \leq M E_x [1 - \exp(-\eta_B)], \quad \forall x \in X, \quad B \in \mathcal{B}(X).$$

Hence if $\varphi \{ B \} > 0$ then

$$(14) \quad E_x [1 - \exp(-\eta_B)] \leq \frac{1}{M} \varphi \{ B \} > 0, \quad x \in C.$$

Now we have assumed that

$$\limsup_{t \rightarrow \infty} \mathbf{1}(\Phi_t \in C) = 1 \quad \text{a.s. } [P_{x_0}],$$

and hence from (14) and Theorem 2.1 (ii),

$$\mathbf{1} \{ \eta_B = \infty \} = \limsup_{t \rightarrow \infty} E_{\Phi_t} [1 - \exp(-\eta_B)] \geq \frac{\varphi \{ B \}}{M} \quad \text{a.s. } [P_{x_0}]$$

which shows that (13) holds for $\Phi_0 = x_0$.

Let H denote the set of all $x \in X$ with the property that the implication (13) holds for every $B \in \mathcal{B}(X)$ when $\Phi_0 = x$. The set H is measurable since the state space is assumed locally compact and separable, and it is easily seen to be absorbing. By

Theorem 2.1 (i) the set H may be expressed

$$H = \{x \in X: \text{for all } B \in \mathcal{B}(X), \varphi\{B\} > 0 \Rightarrow \check{L}(x, B) = 1\}.$$

The set is maximally absorbing by Proposition 2.2 (iii), and hence it is a maximal Harris set by definition. Result (ii) follows from (i) and the fact that maximal Harris sets are disjoint.

As a corollary of this result we see that compact sets can be used to characterize Harris recurrence for irreducible T-processes. We note that Gettoor in [9] also finds conditions under which compact sets characterize recurrence. His condition LSC seems intrinsically much harder to verify than our T-process condition.

With these results on petite sets we are now able to give the following proof.

Proof of Theorem 2.2.

(ii) Let C denote a φ_b -petite set of positive π_0 -measure. Such a set exists by the Harris property assumed for the K_a -chain. In fact, the probability b may be taken to be a^{k*} , the k -fold convolution of the probability a , for some integer k .

By the Harris property,

$$\limsup_{t \rightarrow \infty} \mathbf{I}\{\Phi_t \in C\} = 1 \quad \text{a.s.}$$

for initial conditions in H_0 . Theorem 4.3 (i) implies that for each $x \in H_0$, there exists a maximal Harris set containing x . As in the proof of Theorem 4.3 (i) we see that each of these Harris sets contains H_0 . Since distinct maximal Harris sets are disjoint, this shows that the maximal Harris set containing H_0 is unique.

(iii) This follows from (ii) and the fact that an invariant probability on H may be explicitly constructed: define π through the formula

$$\pi\{B\} = \int_0^\infty \mathbf{E}_{\pi_0}[\mathbf{I}\{\Phi_t \in B\}a[t, \infty)] dt, \quad B \in \mathcal{B}(X)$$

where π_0 is the invariant probability which is supported on H_0 .

Since a has a finite mean it follows that π is a finite measure. Invariance is a consequence of the following identities: for any $t > 0$ and any $B \in \mathcal{B}(X)$ we have by the Markov property

$$\begin{aligned} \pi P^t\{B\} &= \int_0^\infty \mathbf{E}_{\pi_0} \left[\int_t^T \mathbf{I}\{\Phi_s \in B\} ds \right] a(dT) \\ &\quad + \int_0^\infty \mathbf{E}_{\pi_0 P^t} \left[\int_0^t \mathbf{I}\{\Phi_s \in B\} ds \right] a(dT). \end{aligned}$$

The second term is equal to $\mathbf{E}_{\pi_0}[\int_0^t \mathbf{I}\{\Phi_s \in B\} ds]$ by invariance of π_0 , which shows that π is invariant for the process.

In [22] a characterization of the positive Harris recurrence property is also developed in terms of hitting times on petite sets. For any timepoint $\delta \geq 0$ and any

set $C \in \mathcal{B}(X)$ define $\tau_C(\delta) \triangleq \delta + \theta^\delta \tau_C$ as the first hitting time on C after δ : here θ^δ is the usual backwards shift operator [5]. Then $E_x[\tau_C(\delta)]$ is (almost) the expected hitting time on C for small δ . The classification we then have in Theorem 1.2 (a) of [22] is as follows.

Theorem 4.4. If Φ is Harris recurrent then Φ is positive Harris recurrent if and only if there exists a closed petite set C such that for some (and then any) $\delta > 0$

$$(15) \quad \sup_{x \in C} E_x[\tau_C(\delta)] < \infty.$$

5. Skeleton chains and a periodic decomposition

The Doebelin decomposition is based on the identification of Harris sets for the resolvent with those for the process Φ . Such an identification is not always possible, and this section is given to defining the situation for the ‘worst case’ when the sampling distribution is lattice.

A typical example of the problems encountered when the skeletons are not irreducible is given by the clock process in (3). Each individual state $x \in S^1$ is a Harris set for the 1-skeleton, with invariant probability $\pi_x = \delta_x$. The 1-skeleton possesses an uncountable number of Harris sets, and from Theorem 2.2 we see that Φ does possess a positive Harris set H , whose unique invariant probability π may be expressed

$$\pi\{B\} = E_{\pi_x} \left[\int_0^1 \mathbf{1}\{\Phi_t \in B\} dt \right];$$

in fact π is simply the uniform distribution on S^1 . From Theorem 2.2 we also see that $H = S^1$, which is strictly larger than any Harris set for these sampled chains. Hence a Harris set for the K_a -chain need not be a Harris set for the process when a is a lattice probability. In this section we consider the skeleton chains in detail in order to indicate exactly the conditions under which skeleton chains are well behaved.

We consider probabilities which are concentrated at a single point $\Delta > 0$. For brevity we will call the skeleton chain $\{\Phi_{\Delta n} : n \in \mathbb{Z}_+\}$ the Δ -chain. The structural results developed here will be used to obtain a number of general ergodic theorems below and in Meyn and Tweedie [25].

Theorem 5.1. Suppose that a positive maximal Harris set H_0 exists for the Δ -chain with corresponding probability π_0 which is P^Δ -invariant. Then there exists a unique $\Delta \geq \lambda \geq 0$, distinct positive Harris sets $\{H_t : 0 < t \leq \lambda\}$, and corresponding invariant probabilities $\{\pi_t : 0 < t \leq \lambda\}$ satisfying $\pi_0 = \pi_\lambda$, $H_0 = H_\lambda$, and

$$(16) \quad H_t \triangleq \{x \in X : P_x\{\Phi_{n\Delta-t} \in H_0 \text{ for some } n \geq 1\} = 1\}$$

$$(17) \quad \pi_t \triangleq \pi_0 P^t, \quad 0 \leq t \leq \lambda.$$

Although strictly speaking H_t is defined only for $t \in [0, \Delta]$, without any restrictions on t in the definition (16) we see that if $t = s \pmod{\Delta}$ then $H_t = H_s$. Without further comment we write H_t and π_t for an arbitrary $t \geq 0$ with the obvious interpretations. In order to prove the theorem we need two lemmas. We first demonstrate that H_t is a maximal Harris set for the Δ -chain.

Lemma 5.1. The set H_t defined in (16) is a maximal Harris set for the Δ -chain, $t \geq 0$. Hence for each $t, s \geq 0$, the sets H_t and H_s are either disjoint or identical.

Proof. Since H_0 is a Harris set and hence absorbing for the Δ -chain,

$$H_t = \{x \in X : P_x\{\Phi_{n\Delta-t} \in H_0 \text{ for infinitely many } n \geq 1\} = 1\}$$

which implies that H_t is maximally absorbing. We now establish the Harris property. Call a bounded measurable function $f : X \rightarrow \mathbb{R}$ K -harmonic if $Kf = f$ on X . It is well known that if an invariant probability π exists, and if $A \in \mathcal{B}(X)$ is π -positive and absorbing, then A is a positive Harris set for the Markov chain with transition function K if and only if every K -harmonic function is constant on A (see the corollary to Proposition 4.2 of [28]). Let f be P^Δ -harmonic, so that $f = \int f d\pi_0$ on H_0 . From the easy facts that (i) $P^t f$ is also P^Δ -harmonic; (ii) $P^{\Delta-t}(x, H_0) = 1$ for any $x \in H_t$, we deduce that

$$f(x) = P^{\Delta-t} P^t f(x) = P^{\Delta-t}(x, H_0) \left(\int P^t f d\pi_0 \right) = \int f d\pi_t, \quad x \in H_t,$$

which shows that H_t is a Harris set.

We now consider the evolution of the Harris sets $\{H_t : t \in \mathbb{R}_+\}$. Recall that we have assumed that the Harris set H_0 is maximal so that $H_0 = \{x \in X : L_\Delta(x, H_0) = 1\}$, where L_Δ is the hitting probability for the Δ -chain. Define the set $G \subset \mathbb{R}$ as

$$(18) \quad G \triangleq \{t \in \mathbb{R} : H_{|t|} = H_0\}.$$

For $t \leq n\Delta$ we have

$$1 = P^{n\Delta}(x, H_0) = \int P^t(x, dy) P^{n\Delta-t}(y, H_0), \quad x \in H_0.$$

This shows that $P^{n\Delta-t}(y, H_0) = 1$ for a.e. $y \in X$ with respect to the measure $P^t(x, \cdot)$, and hence

$$(19) \quad P^t(x, H_t) = 1, \quad t \geq 0, \quad x \in H_0.$$

Since distinct maximal Harris sets are disjoint and their invariant probabilities are mutually singular, (19) implies that

$$G = \{t \in \mathbb{R} : P^{|t|}(x, H_0) = 1, x \in H_0\} = \{t \in \mathbb{R} : \pi_0 P^{|t|} = \pi_0\}.$$

Lemma 5.2. The set G is a subgroup of the additive group \mathbb{R} . For each $s \in \mathbb{R}$, the Harris sets $\{H_t : t \in \{G + \{s\}\} \cap \mathbb{R}_+\}$ are identical.

Proof. Let $s, t \in G$, and assume that $|t| > |s|$. Then either $|t + s| = |t| - |s|$, or $|t + s| = |t| + |s|$. In the first case

$$\pi_0 P^{|t+s|} = \pi_0 P^{|s|} P^{|t|-|s|} = \pi_0 P^{|t|} = \pi_0$$

while in the second case

$$\pi_0 P^{|t+s|} = \pi_0 P^{|t|} P^{|s|} = \pi_0 P^{|s|} = \pi_0$$

which shows that G is a group.

We now prove the second assertion. Let $r_1, r_2 \in \{G + \{s\}\} \cap \mathbb{R}_+$, with $r_2 > r_1$. Then the chain of equalities

$$\pi_{r_1} \triangleq \pi_0 P^{r_1} = \pi_0 P^{r_2-r_1} P^{r_1} = \pi_0 P^{r_2} = \pi_{r_2}$$

shows that $H_{r_1} = H_{r_2}$.

Proof of Theorem 5.1. Let $\lambda = \inf\{t > 0 : t \in G\}$. If $G = \mathbb{R}$ or $\lambda > 0$ then the conclusions of the theorem follow from Lemma 5.1 and Lemma 5.2. We now show that no alternative is possible. Suppose on the contrary that G is a dense, proper subset of \mathbb{R} , and let $s \in G^c \cap \mathbb{R}_+$. Since H_s and H_0 are disjoint, the invariant probabilities $\pi_s = \pi_0 P^s$ and π_0 are mutually singular. Hence there exists a compact set $F \subset X$ satisfying

$$(20) \quad \pi_0\{F\} = 0 \quad \text{and} \quad \pi_s\{F\} > 0.$$

If $\Phi_0 = x \in F^c$ then, since Φ has right-continuous sample paths, $\tau_F > 0$ a.s. $[\mathbf{P}_x]$. Hence for such x ,

$$\lim_{t \downarrow 0} P^t(x, F) \leq \lim_{t \downarrow 0} \mathbf{P}_x\{\tau_F \leq t\} = 0.$$

Since $\pi_0\{F\} = 0$, the dominated convergence theorem implies that

$$\lim_{t \downarrow 0} \pi_0 P^t\{F\} = 0.$$

However, by Lemma 5.2,

$$\pi_0 P^t\{F\} = \pi_s\{F\} > 0, \quad t \in \mathbb{R}_+ \cap \{G + \{s\}\}.$$

Since the set $\{G + \{s\}\}$ is dense as a subset of \mathbb{R} , this is the desired contradiction.

Theorem 5.1 shows that the initial distribution π_0 gives rise to a periodic orbit of probabilities with period $\lambda \geq 0$. Hence in some sense the clock process (3) discussed at the beginning of this section exhibits the only type of pathology which is possible when considering the skeleton chain.

We now show that we do not need irreducibility to counteract the pathology: it is excluded even when the skeleton chain possesses a countable recurrence structure.

Theorem 5.2. Suppose that the Δ -chain possesses an at most countable collection of Harris sets. Then the following statements are valid for any positive maximal

Harris set H_0 for the Δ -chain:

- (i) The period λ defined in Theorem 5.1 is equal to zero;
- (ii) The Δ -chain restricted to H_0 is aperiodic;
- (iii) The invariant probabilities π and π_0 coincide.

Proof. (i) follows from Theorem 5.1, for if $\lambda > 0$, then the Harris sets $\{H_t : 0 < t \leq \lambda\}$ are distinct and uncountable. Result (ii) follows from (i) since, by definition, H_0 is aperiodic if it is a Harris set for the $d\Delta$ -chain for each $d \geq 1$. (iii) follows immediately from the representation for π in Theorem 2.2.

We shall use this structure in the final section of the paper, where we establish ergodic theorems for continuous-time T-processes which are not necessarily irreducible.

6. Ergodic theorems for irreducible processes

In discrete time, positive Harris recurrence is equivalent to total variation norm convergence of the distributions of the chain [26], at least in Cesaro average.

In the countable-state continuous-time cases, positive Harris recurrence implies total variation convergence and also aperiodicity, so that Cesaro convergence is not needed. In the general case, Harris recurrence seems to be too weak to imply such strong convergence of the distributions of Φ : a counterexample is provided by the clock process, as given in (3) in Section 5, although the distributions in this example do converge over an uncountable cycle for this deterministic motion. Such a form of Cesaro convergence has been recently proved to hold in general by Glynn and Sigman [10]. Here we investigate cases where the Cesaro-type averaging is not needed. The Markov process Φ will be called *ergodic* if an invariant probability π exists and

$$\lim_{t \rightarrow \infty} \|P^t(x, \cdot) - \pi\| = 0, \quad \forall x \in X.$$

If Φ is ergodic, then it follows immediately that *every* skeleton chain with transition kernel P^δ is also ergodic. Several typical positive Harris recurrent processes do not have this property (again take the clock process (3)) and hence some additional conditions are required to obtain ergodicity.

This example shows that a Harris set for the K_a -chain need not be a Harris set for the process when a is lattice. This is unfortunate since to obtain total variation norm limit theorems for the process, it is clearly *necessary* that the Harris sets agree.

One route to proving the desired convergence results involves, for example, an assumption of continuity in t of the probabilities $P^t(x, A)$, as in [32], [30]; this then

enables us to prove that the skeleton chains have the same structure as the process.

In this section our approach is to assume only that some one skeleton chain is irreducible. Under this one condition, we now show that positive Harris recurrence and ergodicity are equivalent. Alternative sets of conditions for ergodicity or a generalization of ergodicity will be given later in this section and in the final section of the paper.

Theorem 6.1. Suppose that Φ is positive Harris recurrent with invariant probability π . Then Φ is ergodic if and only if some skeleton chain is irreducible.

Proof. The necessity is obvious. To see sufficiency, suppose the skeleton $\{\Phi_{k\Delta}\}$ is irreducible, and since π is necessarily invariant for this skeleton, it follows that the skeleton is positive recurrent, and hence possesses a Harris set $H \subset X$ (cf. Theorem 3.7 of [26]).

It is well known that when π is an invariant probability, the total variation norm $\|P^t(x, \cdot) - \pi\|$ is a decreasing function of $t \in \mathbb{R}_+$ [30]. Hence to prove the result it is sufficient to show that a subsequence converges to zero. This is clearly the case for x in H provided the Harris-recurrent chain on H is aperiodic [26]: the point of this proof is to show firstly that convergence occurs for all x , and secondly that aperiodicity holds.

Let $Q(x) = P_x\{\Phi_{n\Delta} \in H \text{ for some } n \in \mathbb{Z}_+\}$, and

$$q(x) = \frac{1}{\Delta} \int_0^\Delta P^s Q(x) ds.$$

Since Q is harmonic for the Δ -chain (cf. [24]) it follows that q is harmonic for the process: that is, $P^t q = q$ for all t . Since Φ is Harris recurrent, it follows that q takes on the constant value

$$q(x) = \int q d\pi = \int Q d\pi = 1$$

for all $x \in X$. By the definition of q it follows that for each $x \in X$, $P^s Q(x) = 1$ for a.e. $s \in \mathbb{R}_+[\mu^{\text{Leb}}]$. This implies that, with $\mu_s = P^s(x, \cdot)$ and $x \in X$ fixed,

$$P_{\mu_s}\{\Phi_{n\Delta} \in H \text{ for some } n \in \mathbb{Z}_+\} = 1.$$

Hence at least Cesaro limits apply for all x . But irreducibility of the skeleton gives us aperiodicity from Theorem 5.2, and so from standard limit theorems for Harris chains,

$$\lim_{n \rightarrow \infty} \|P^{n\Delta+s}(x, \cdot) - \pi\| = \|\mu_s P^{n\Delta} - \pi\| = 0$$

which completes the proof.

Results linking ergodic results of the process and its skeletons are well known for countable space processes, but the approach relies heavily on the continuity (in t) of the transition probability functions.

In a general setting, in [32] it is shown (also under some continuity conditions in t on the semigroup P^t) that ergodicity of the process Φ follows from the ergodicity of the embedded skeletons or of the resolvent chains; whilst related results using ‘regenerative’ sets are also stated in Chapter VI.3 of [1]. However, the conditions under which the ‘regeneration’ can be guaranteed to take place (such as when (3.1) of [1], p. 150 is satisfied for an interval of time values) are not always obvious.

We now show that under the conditions of the theorem, petite sets are equivalent to small sets, as defined in [26]. This result will simplify considerably the proofs below, but it has considerable independent interest, in that it shows that the Nummelin splitting of the process at some future timepoint is possible.

Proposition 6.1. Suppose that Φ is positive Harris recurrent, and that some skeleton chain is irreducible. If C is petite, then there exists a constant $T > 0$ and a non-trivial measure μ such that

$$P^s(x, \cdot) \geq \mu\{\cdot\}, \quad s \geq T, \quad x \in C.$$

Proof. Suppose that C is ψ_a -petite. By Theorem 6.1, Egorov’s theorem and the fact that $\|P^t(x, \cdot) - \pi\{\cdot\}\|$ is decreasing in t , there exists a set $C_0 \in \mathcal{B}(X)$ with the property

$$(21) \quad \limsup_{t \rightarrow \infty} \sup_{x \in C_0} \|P^t(x, \cdot) - \pi\{\cdot\}\| = 0.$$

We can and will assume that C_0 is closed, and that $C_0 \in \mathcal{B}^+(X)$. From ergodicity for the unit-time skeleton there exists $n \geq 1$, a set B with $\pi\{B\} > 0$ and a measure φ_n such that $P^n(y, \cdot) \geq \varphi_n\{\cdot\}$ for $x \in B$ (cf. Theorem 2.1 of [26]). It follows that C_0 is $\varphi_{\delta_{n_0}}$ -petite for some integer $n_0 > 0$: we have

$$(22) \quad \begin{aligned} P^{m+n}(x, \cdot) &\geq \int_B P^m(x, dy) P^n(y, \cdot) \geq \varphi_n(\cdot) P^m(x, B) \\ &\geq \varphi_n(\cdot) \pi(B)/2 \\ &\triangleq \varphi_{\delta_{m+n}}\{\cdot\} \end{aligned}$$

independent of $x \in C_0$ for all large enough m , from (21). For any $t > 0$ we have the estimate

$$\begin{aligned} P^t(x, C_0) &\geq \int_0^{t/2} \left\{ \int_{C_0} P^s(x, dy) P^{t-s}(y, C_0) \right\} a(ds) \\ &\geq \int_0^{t/2} P^s(x, C_0) a(ds) \left(\inf_{r \geq t/2} \inf_{y \in C_0} P^r(y, C_0) \right). \end{aligned}$$

Since C is ψ_a -petite and C_0 satisfies the uniformity property (21), we have for all t

sufficiently large

$$P^t(x, C_0) \geq \varepsilon_0 \triangleq (\psi_a\{C_0\}/2)(\pi\{C_0\}/2) > 0, \quad x \in C.$$

Since C_0 is $\varphi_{\delta_{n_0}}$ -petite, this shows that $P^{t+n_0}(x, \cdot) \geq \varepsilon_0 \varphi_{\delta_{n_0}}(\cdot)$ for all $x \in C$, and all t sufficiently large, which completes the proof.

This result implies that when Φ is ergodic and all compact subsets of X are petite, then for every skeleton chain all compact subsets of X are petite.

7. f -ergodic theorems

Theorem 6.1 presents conditions under which the expectation of bounded functions of the process converge to a steady state value for all initial conditions.

Recent advances in discrete-time chains have led to the generalization of this result to unbounded functions (cf. [24]): for this we need the concept of the f -norm $\|\mu\|_f$. For any positive measurable function $f \geq 1$ and any signed measure μ on $\mathcal{B}(X)$ we write

$$\|\mu\|_f = \sup_{|g| \leq f} |\mu(g)|.$$

Note that the total variation norm $\|\mu\|$ is $\|\mu\|_f$ in the special case where $f \equiv 1$.

For a measurable function $f \geq 1$ we call Φ f -ergodic if it is positive Harris recurrent with invariant probability π , if $\pi(f) < \infty$, and

$$\lim_{t \rightarrow \infty} \|P^t(x, \cdot) - \pi\|_f = 0, \quad \forall x \in X.$$

There is a generalization of Theorem 4.4 which guarantees that $\pi(f)$ is finite and which we shall use in verifying f -ergodicity. Recall that $\tau_C(\delta) \triangleq \delta + \theta^d \tau_C$ is the first hitting time on C after δ . The kernel $G_C(x, A; \delta)$ is defined for any x and measurable A through

$$(23) \quad G_C(x, A; \delta) \triangleq \mathbf{E} \left[\int_0^{\tau_C(\delta)} \mathbf{1}\{\Phi_t \in A\} dt \right],$$

so that in particular for the choice of $A = X$

$$G_C(x, X; \delta) = \mathbf{E}_x[\tau_C(\delta)]$$

which is used in classifying chains as positive Harris recurrent in Theorem 4.4. Following discrete-time usage [26], [23] a non-empty set $C \in \mathcal{B}(X)$ is called f -regular if

$$G_B(x, f; \delta) = \mathbf{E}_x \left[\int_0^{\tau_B(\delta)} f(\Phi_t) dt \right]$$

is bounded on C for any $\delta > 0$ and any $B \in \mathcal{B}^+(X)$.

Theorem 7.1. If Φ is Harris recurrent with invariant measure π and $f \geq 1$ is a measurable function on X , then the following are equivalent:

(i) There exists a closed petite set C such that for some (and then any) $\delta > 0$

$$(24) \quad \sup_{x \in C} G_C(x, f; \delta) < \infty;$$

(ii) Φ is positive Harris recurrent and $\pi(f) < \infty$.

When (24) holds the set C is f -regular.

Proof. Use Theorem 1.2(b) and Proposition 4.1 of [22].

Even when $\pi(f) < \infty$, for unbounded f we have not been able to obtain a relationship between f -ergodicity of Φ and its skeletons without making somewhat stronger assumptions on the process. One condition which provides such a connection, and hence allows a proof of f -norm convergence for the distributions of the process, is the following growth condition on the expectation of $f(\Phi_t)$ over intervals near zero: for some constant $\delta > 0$, and a measurable function $h \geq 1$,

$$(25) \quad P^s f \leq h, \quad 0 \leq s \leq \delta.$$

Theorem 7.2. Suppose that for the Markov process Φ , some skeleton chain is irreducible. Let $f, h \geq 1$ satisfy (25), and suppose that a closed h -regular set C exists with $G_C(x, h; 0) < \infty$ for all $x \in X$. Then Φ is f -ergodic.

Proof. By Theorem 4.3, Theorem 7.1 and Theorem 6.1 the process together with each of its skeleton chains is positive Harris recurrent and $\pi(h) < \infty$, where as usual π denotes the invariant probability for the process. From Theorem 15.0.1 of [23] and Egorov's theorem, there exists a closed set $A_0 \in \mathcal{B}^+(X)$ for which

$$(26) \quad \limsup_{n \rightarrow \infty} \sup_{x \in A_0} \|P^{n\delta}(x, \cdot) - \pi\{\cdot\}\|_h = 0.$$

Since $A_0 \in \mathcal{B}^+(X)$ we have by Proposition 4.2 (ii) of [22] that

$$(27) \quad E_x \left[\int_0^{\tau_{A_0}} h(\Phi_s) ds \right] < \infty, \quad \forall x \in X.$$

Then writing $t = n\delta + s$ and using the bound (25) on $P^s f$ and (26) we have for any $|g| \leq f$, and any $x \in A_0$,

$$\begin{aligned} |P^t(x, g) - \pi(g)| &= |P^{n\delta}(x, P^s g) - \pi(P^s g)| \\ &\leq \|P^{n\delta}(x, \cdot) - \pi\{\cdot\}\|_h \rightarrow 0, \quad \text{as } t \rightarrow \infty \end{aligned}$$

where the convergence is uniform for $x \in A_0$ and $|g| \leq f$. That is,

$$(28) \quad \limsup_{t \rightarrow \infty} \sup_{x \in A_0} \|P^t(x, \cdot) - \pi\{\cdot\}\|_f = 0.$$

For arbitrary $x \in X$ we may estimate as follows. Letting $\bar{g} = g - \pi(g)$ we have for

$$|g| \leq f,$$

$$(29) \quad |P^t(x, \bar{g})| \leq \int_0^t \sup_{y \in \bar{A}_0} |P^{t-s}(y, \bar{g})| \mathbf{P}_x\{\tau_{A_0} \in ds\} + |\mathbf{E}_x[\bar{g}(\Phi_t)\mathbf{I}\{\tau_{A_0} > t\}]|.$$

We see from (28) and (27) that the first term converges to zero as $t \rightarrow \infty$, uniformly in $|g| \leq f$. The second term is bounded by $\mathbf{E}_x[(f(\Phi_t) + \pi(f))\mathbf{I}\{\tau_{A_0} > t\}]$, which we can bound for any $0 \leq r \leq \delta$, and all $t \geq \delta$, by

$$\begin{aligned} \mathbf{E}_x[(f(\Phi_t) + \pi(f))\mathbf{I}\{\tau_{A_0} > t\}] &\leq \mathbf{E}_x[(f(\Phi_t) + \pi(f))\mathbf{I}\{\tau_{A_0} > t - r\}] \\ &= \mathbf{E}_x[(P^r f(\Phi_{t-r}) + \pi(f))\mathbf{I}\{\tau_{A_0} > t - r\}] \\ &\leq \mathbf{E}_x[(h(\Phi_{t-r}) + \pi(h))\mathbf{I}\{\tau_{A_0} > t - r\}]. \end{aligned}$$

Under the conditions of the theorem, the right-hand side of the inequality is an integrable function of t . It follows that

$$\begin{aligned} &\limsup_{t \rightarrow \infty} \mathbf{E}_x[(f(\Phi_t) + \pi(f))\mathbf{I}\{\tau_{A_0} > t\}] \\ &\leq \limsup_{t \rightarrow \infty} \inf_{0 \leq r \leq \delta} \mathbf{E}_x[(h(\Phi_{t-r}) + \pi(h))\mathbf{I}\{\tau_{A_0} > t - r\}] \\ &= 0. \end{aligned}$$

Hence the distributions converge in f -norm, as we wanted.

Typically Theorem 7.2 is applied with $h = c_\delta f$ for some constant c_δ . However the extra generality is sometimes useful, as is seen in the following corollary.

Proposition 7.1. Suppose that for the Markov process Φ , some skeleton chain is irreducible, and that a closed f -regular set C exists with $G_C(x, f; 0) < \infty$ for all $x \in X$. For any $\Delta > 0$ define the function f_Δ by

$$f_\Delta(x) = \int_0^\Delta P^s f(x) ds.$$

Then we have

$$\lim_{t \rightarrow \infty} \|P^t(x, \cdot) - \pi\|_{f_\Delta} = 0, \quad \forall x \in X.$$

Proof. Let $t = \Delta + \delta$ where $\delta > 0$ is arbitrary. By the Markov property, we have for all $s \leq t$,

$$\begin{aligned} \mathbf{E}_x \left[\int_0^{\tau_C(t)} P^s f(\Phi_u) du \right] &= \mathbf{E}_x \left[\int_0^{\tau_C(t)} f(\Phi_{u+s}) du \right] \\ &= \mathbf{E}_x \left[\int_s^{\tau_C(t)} f(\Phi_u) du \right] \\ &\quad + \mathbf{E}_x \left[\int_{\tau_C(t)}^{\tau_C(t)+s} f(\Phi_u) du \right]. \end{aligned}$$

Integrating both sides of this bound for $s = 0$ to $\Delta + \delta$ we obtain the bound

$$(30) \quad E_x \left[\int_0^{\tau_C(t)} f_{\Delta+\delta}(\Phi_u) du \right] \leq (\Delta + \delta) E_x \left[\int_0^{\tau_C(t)} f(\Phi_u) du \right] + (\Delta + \delta)k$$

where, by the strong Markov property and the fact that $\Phi_{\tau_C(t)} \in C$,

$$k \triangleq \sup_{x \in C} E_x \left[\int_{\tau_C(t)}^{\tau_C(t)+t} f(\Phi_u) du \right] \leq \sup_{x \in C} E_x \left[\int_0^t f(\Phi_u) du \right] < \infty.$$

From the assumptions of the theorem, the right-hand side of (30) is finite for each x , and uniformly bounded for $x \in C$. Letting $h = f_{\Delta+\delta}$, we also have $P^s f_{\Delta} \leq h$ for all $s \leq \delta$, so that (25) holds. Hence the result follows from Theorem 7.2.

8. Decomposable T-processes

We conclude with the extension of the results above to the case where the chain is not necessarily irreducible. If Φ is a T-process which is bounded in probability on average then almost every trajectory enters some maximal positive Harris set, as shown in the decomposition theorem. Let π_i denote the unique invariant probability which is supported on H_i , let $C_i \subset H_i$ denote closed sets of positive π_i -measure, and let \tilde{H}_i denote the event that Φ enters C_i , $i \in I$. The events $\{\tilde{H}_i : i \in I\}$ are disjoint, and the probability of their union is equal to 1: this follows from Theorem 8.1 (i) by setting $f = \mathbf{1}_{H_i}$, $i \in I$.

We define the random probability $\tilde{\pi}$ as

$$\tilde{\pi}\{B\} \triangleq \sum_{i \in I} \mathbf{1}\{\tilde{H}_i\} \pi_i\{B\}, \quad B \in \mathcal{B}(X).$$

The invariant transition function Π is defined as $\Pi(x, B) \triangleq E_x[\tilde{\pi}\{B\}]$. From Theorem 8.1 we see that $P\Pi = \Pi P = \Pi$.

These quantities describe the limiting behavior of the distributions, and the occupation probabilities of the Markov process Φ , as was the case in discrete-time chains [24].

Theorem 8.1. Suppose that Φ is bounded in probability on average. Then

(i) If Φ is a T-process, then for any $x \in X$ and any $f \in L_1(X, \mathcal{B}(X), \Pi(x, \cdot))$,

$$\lim_{r \rightarrow \infty} \frac{1}{r} \int_0^r f(\Phi_t) dt = \int f d\tilde{\pi} \quad \text{a.s. } [P_*].$$

(ii) Suppose that for a lattice distribution a , the kernel K_a possesses an everywhere non-trivial continuous component. Then for every $x \in X$,

$$\lim_{t \rightarrow \infty} \|P^t(x, \cdot) - \Pi(x, \cdot)\| = 0.$$

(iii) If the hypotheses of (ii) hold and the state space X is compact, then there exists $\rho < 1$ such that

$$\limsup_{t \rightarrow \infty} \rho^{-t} \|P^t(x, \cdot) - \Pi(x, \cdot)\| = 0.$$

Proof. (i) This result holds for each initial condition lying in a Harris set by the ergodic theorem for Harris-recurrent Markov processes ([3], p. 169). To prove the result for arbitrary initial conditions, we may adapt the proof of Theorem 5.5 of [24] by conditioning at the first entrance time to $\sum C_i$ (where C_i are the compact sets used in defining $\tilde{\pi}$) and applying the strong Markov property.

(ii) The proof of this result is essentially the same as in the discrete-parameter case in [24], and we omit the details.

(iii) If X is compact, then from Theorem 5.5 of [24] and the fact that each Harris set is aperiodic we have for some $\rho < 1$,

$$(31) \quad \limsup_{k \rightarrow \infty} \rho^{-\Delta k} \|P^{\Delta k}(x, \cdot) - \Pi(x, \cdot)\| = 0.$$

Again using the fact that $\|P^t(x, \cdot) - \Pi(x, \cdot)\|$ is decreasing, Equation (31) implies that

$$\rho^{-t} \|P^t(x, \cdot) - \Pi(x, \cdot)\| \leq \rho^{-\Delta} \rho^{-\Delta k} \|P^{\Delta k}(x, \cdot) - \Pi(x, \cdot)\|$$

whenever $t - \Delta < \Delta k \leq t$, proving (iii).

Since we have shown in Section 3.3 that hypoelliptic diffusions are T-processes, result (i) provides a different method of proof of the law of large numbers given on p. 705 of Kliemann [17]. Note that the actual statement of that result is incorrect: the random probability $\tilde{\pi}\{B\}$ should be used in place of Π used there, as in (i) rather than (ii).

Finally, we use arguments such as the above to complete the cycle of equivalences between topological and probabilistic stability conditions in the reducible case.

Theorem 8.2. Suppose that Φ is a T-process. Then

(i) Φ is non-evanescent if and only if

$$X = \sum_{i \in I} H_i + E = H + E$$

where each H_i is a maximal Harris set, and $P_x\{\eta_H = \infty\} \equiv 1$.

(ii) Φ is bounded in probability on average if and only if it is non-evanescent, and every Harris set in H is positive.

Proof. (i) This follows directly from (iii) of the decomposition theorem.

(ii) If Φ is a T-process and bounded in probability on average, then it is non-evanescent, and it follows from (iv) of the decomposition theorem that every Harris set is positive.

Suppose now that Φ is non-evanescent, and that every Harris set is positive.

Using (iii) of the decomposition theorem, we may adapt the proof of Theorem 8.1 (i) to show that for any $x \in X$ and any bounded measurable function $f : X \rightarrow \mathbb{R}$,

$$\lim_{r \rightarrow \infty} \frac{1}{r} \int_0^r f(\Phi_t) dt = \int f d\bar{\pi} \quad \text{a.s. } [P_*]$$

where

$$\bar{\pi}\{B\} \triangleq \sum_{i \in I} \mathbf{1}\{\Phi \text{ enters } C_i\} \pi_i\{B\} \quad B \in \mathcal{B}(X)$$

and each $C_i \subset H_i$ is compact with positive π_i -measure. Since

$$E_x \left[\sum_{i \in I} \mathbf{1}\{\Phi \text{ enters } C_i\} \right] = 1$$

for all x , it follows that Φ is bounded in probability on average.

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