Generalized Resolvents and Harris Recurrence of Markov Processes *

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Abstract

In this paper we consider a φ-irreducible continuous parameter Markov process $\mathbf{\Phi}$ whose state space is a general topological space. The recurrence and Harris recurrence structure of $\mathbf{\Phi}$ is developed in terms of generalized forms of resolvent chains, where we allow state-modulated resolvents and embedded chains with arbitrary sampling distributions. We show that the recurrence behavior of such generalized resolvents classifies the behavior of the continuous time process; from this we prove that hitting times on the small sets of a generalized resolvent chain provide criteria for, successively, (i) Harris recurrence of $\mathbf{\Phi}$ (ii) the existence of an invariant probability measure π (or positive Harris recurrence of $\mathbf{\Phi}$) and (iii) the finiteness of $\pi(f)$ for arbitrary f.

KEYWORDS: Irreducible Markov processes, ergodicity, exponential ergodicity, recurrence, resolvents.

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1 Introduction

The stability and ergodic theory of continuous time Markov processes has a large literature which includes many different approaches. One such is through the use of associated discrete time "resolvent chains". The recurrence structure of the process and that of the resolvent chain are essentially equivalent [1] and, since the analysis of discrete time chains is well understood [20], this result simplifies the analysis considerably.

In this paper we develop two generalized forms of the resolvent (1), and isolate a specific type of subset of the state space (a "petite set") such that the behavior of the generalized resolvent chains on such a set provides criteria for Harris recurrence and positive Harris recurrence. This extends the approach in [23], where continuity conditions were needed on the transition probabilities of the chain to achieve similar results.

We suppose that $\mathbf{\Phi} = \{\Phi_t : t \in \mathbb{R}_+\}$ is a time homogeneous Markov process with state space $(\mathsf{X}, \mathcal{B})$, and transition semigroup (P^t). For each initial $\Phi_0 = x \in \mathsf{X}$, the process $\mathbf{\Phi}$ evolves on the probability space $(\Omega, \mathcal{F}, \mathsf{P}_x)$, where Ω denotes the sample space. Further details of this framework may be found in [22].

It is assumed that the state space X is a locally compact separable metric space, and that $\mathcal{B}$ is the Borel field on X. We assume that $\mathbf{\Phi}$ is a Borel right process, so that in particular $\mathbf{\Phi}$ is strongly Markovian with right continuous sample paths [22]. When an event $\mathcal{A}$ in sample space holds almost surely for every initial condition we shall write "$\mathcal{A}$ holds a.s. $[\mathsf{P}_*]$".

The operator P^t acts on bounded measurable functions f and σ-finite measures μ on X via

$$P^t f\,(x) = \int_{\mathrm{X}} P^t(x, dy) f(y) \qquad \mu P^t\,(A) = \int_{\mathrm{X}} \mu(dx) P^t(x, A).$$

The *resolvent* for the process is defined as

$$R(x, A) \triangleq \int_0^\infty e^{-t} P^t(x, A)\, dt, \quad x \in \mathsf{X}, A \in \mathcal{B}. \tag{1}$$

For a measurable set A we let

$$\tau_A = \inf\{t \geq 0 : \Phi_t \in A\}, \qquad \eta_A = \int_0^\infty \mathbf{1}\{\Phi_t \in A\}\, dt.$$

A Markov process is called *φ-irreducible* if for the σ-finite measure φ,

$$\varphi\{B\} > 0 \Longrightarrow \mathsf{E}_x[\eta_B] > 0, \qquad x \in \mathsf{X}.$$

As in the discrete time setting, if $\mathbf{\Phi}$ is φ-irreducible then there exists a *maximal* irreducibility measure ψ such that $\nu \prec \psi$ for any other irreducibility measure ν [20]. We shall reserve the symbol ψ for such a maximal irreducibility measure, and we will let $\mathcal{B}^+$ denote the collection of all measurable subsets $A \subset \mathsf{X}$ such that $\psi(A) > 0$. We will say that A is *full* if $\psi(A^c) = 0$.

Suppose that, for some σ-finite measure φ, the event $\{\eta_A = \infty\}$ holds a.s. $[\mathsf{P}_*]$ whenever $\varphi\{A\} > 0$. Then $\mathbf{\Phi}$ is called *Harris recurrent*: this is the standard definition of Harris recurrence, which is taken from [2]. Clearly a Harris recurrent chain is φ-irreducible.

Using two different forms of generalized resolvent we will show that Harris recurrence is equivalent to a (formally) much weaker and more useful criterion, and we will derive a criterion for Harris recurrence in terms of *petite sets* defined in Section 3. The following summarizes the results proved in Theorem 2.4, Proposition 3.4, and Theorem 3.3:

Theorem 1.1 *The following are equivalent:*

(i) *The Markov chain $\mathbf{\Phi}$ is Harris recurrent;*

(ii) *there exists a σ-finite measure μ such that $\mathsf{P}_x\{\tau_A < \infty\} \equiv 1$ whenever $\mu\{A\} > 0$;*

(iii) *there exists a petite set C such that $\mathsf{P}_x\{\tau_C < \infty\} \equiv 1$.*

A σ-finite measure π on $\mathcal{B}$ with the property

$$\pi\{A\} = \pi P^t\{A\} \triangleq \int \pi(dx) P^t(x, A) \qquad A \in \mathcal{B},\ t \geq 0$$

will be called *invariant*. It is shown in [8] that if $\mathbf{\Phi}$ is a Harris recurrent right process then an essentially unique invariant measure π exists (see also [2]). If the invariant measure is finite, then it may be normalized to a probability measure, and in practice this is the main situation of interest. If $\mathbf{\Phi}$ is Harris recurrent, and π is finite, then $\mathbf{\Phi}$ is called *positive Harris recurrent*.

Again through the use of generalized resolvents, we find conditions under which π is finite, and indeed for which $\pi(f) < \infty$ for general functions f. These involve expected hitting times on petite sets, but because of the continuous time parameter of the process we have to be careful in defining such times.

For any timepoint $\delta \geq 0$ and any set $C \in \mathcal{B}$ define $\tau_C(\delta) \triangleq \delta + \theta^\delta \tau_C$ as the first hitting time on C after δ: here θ^δ is the usual backwards shift operator [22]. The kernel $G_C(x, f; \delta)$ is defined for any x and positive measurable function f through

$$G_C(x, f; \delta) \triangleq \mathsf{E}\Big[\int_0^{\tau_C(\delta)} f(\Phi_t)\, dt\Big], \tag{2}$$

so that in particular for the choice of $f \equiv 1$

$$G_C(x, \mathsf{X}; \delta) = \mathsf{E}_x[\tau_C(\delta)]$$

is (almost) the expected hitting time on C for small δ. The classification we then have is

Theorem 1.2 *If $\mathbf{\Phi}$ is Harris recurrent with invariant measure π then*

(a) *$\mathbf{\Phi}$ is positive Harris recurrent if and only if there exists a closed petite set C such that for some (and then any) $\delta > 0$*

$$\sup_{x \in C} \mathsf{E}_x[\tau_C(\delta)] < \infty; \tag{3}$$

(b) *if $f \geq 1$ is a measurable function on X, then the following are equivalent:*

(i) *There exists a closed petite set C such that*

$$\sup_{x\in C} G_C(x,f;\delta) < \infty$$

for some (and then any) $\delta > 0$;

(ii) $\mathbf{\Phi}$ *is positive Harris recurrent and $\pi(f) < \infty$.*

Part (a) of this theorem is a special case of part (b), with $f \equiv 1$; and both are proved in Section 4.

The identification of petite sets is therefore important, and in [17] we show that under suitable continuity conditions on the generalized resolvents, all compact sets are petite. These are much weaker than those in the literature, even for special classes of processes, and they certainly hold if the resolvent has the strong Feller property that $R(x,f)$ is continuous if f is bounded. In the special case of diffusion processes on manifolds, necessary and sufficient conditions under which the resolvent of the process possesses the strong Feller property have been obtained in [4] (see also [10, 3, 11, 12]). Strong Feller processes are however a relatively restricted class of processes. Under the condition that the excessive functions for the resolvent-chain are lower semi continuous, characterizations of recurrence are also obtained in terms of hitting probabilities to compact subsets of the state space in [8]: these are similar to those we find for petite sets, but the conditions for petite sets to be compact in [17] appear much weaker than the conditions used in [8].

Thus the results presented here unify and subsume many somewhat diverse existing approaches to recurrence structures for Markov processes.

In [17, 18] we use these results in a number of ways, giving not only the characterization of compact sets as petite sets, but also developing a Doeblin decomposition for non-irreducible chains, verifiable characterizations for ergodicity and rates of convergence, and conditions for convergence of the expectations $\mathsf{E}[f(\Phi_t)]$ for unbounded f.

2 State-modulated Resolvents & Harris Recurrence

The central idea of this paper is to consider the Markov process sampled at times $\{T(k) : k \in \mathbb{Z}_+\}$. These times will sometimes form an undelayed renewal process which is independent of the Markov process $\mathbf{\Phi}$, or a sequence of randomized stopping times. In either case, the sequence $\{T(k)\}$ will be constructed so that the process $\{\Phi_{T(k)}\}$ is a Markov chain evolving on X, whose recurrence properties under appropriate conditions will be shown to be closely related to those of the original process.

This extends the now-classical form of analysis using the resolvent for the process. The kernel $R(x,A)$ in (1) is clearly such a Markov transition function, with transitions given by sampling the process at points of a Poisson process of unit rate. The Markov chain $\check{\mathbf{\Phi}}$ with transition function R will be called the *R-chain*. If $\check\tau_A$ denotes the first return time to the set A for the R-chain, we shall let $\check L(x,A) = \mathsf{P}_x(\check\tau_A < \infty)$ denote the hitting probability for the R-chain, $\check G = \sum R^n$ its potential kernel, and we set

$$\check G_B(x,A) \triangleq \mathsf{E}_x\Big[\sum_{k=1}^{\check\tau_B} \mathbf{1}_A(\check\Phi_k)\Big]$$

For any fixed constant $\alpha > 0$ let T_α be a random time which is independent of the process $\mathbf{\Phi}$ with an exponential distribution having mean α^{-1}. We define

$$R_\alpha(x, A) = \int_0^\infty \alpha e^{-\alpha t} P^t(x, A)\, dt,$$

so that the transition function R_α has the interpretation

$$R_\alpha(x, A) = \mathsf{P}_x\{\Phi_{T_\alpha} \in A\}. \tag{4}$$

If we set

$$U_\alpha(x, A) = \int_0^\infty e^{-\alpha t} P^t(x, A)\, dt,$$

then U_α also has the probabilistic interpretation

$$U_\alpha(x, A) = \mathsf{E}_x\Big[\int_0^{T_\alpha} \mathbf{1}(\Phi_t \in A)\, dt\Big] \tag{5}$$

When $\alpha = 1$, of course both of these expressions coincide and are equal to $R(x, A)$.

We will consider Markov chains derived from $\mathbf{\Phi}$, in two different ways which extend the idea of resolvent chains. The definitions of irreducibility, Harris recurrence, and positive Harris recurrence have exact analogues for such discrete parameter chains. See [15, 20, 21] for these concepts.

In this section, the definition of the chain R_α derived by exponentially sampling at rate α is first generalized so that α, the rate of occurrence of the "sampling time" T_α, may depend upon the value of the state Φ_t. When the rate function is appropriately defined the random time T_α becomes a randomized stopping time for the process. This gives rise to a class of kernels introduced by Neveu in [19] which allow detailed connections between the recurrence structure of a Markov process and the generalized resolvent chains.

Let h be a bounded non-negative measurable function on X, and define the kernels R_h, U_h by

$$R_h(x, A) \triangleq \mathsf{E}_x\Big[\int_0^\infty \exp\Big\{-\int_0^t h(\Phi_s)\, ds\Big\} h(\Phi_t) \mathbf{1}_A(\Phi_t)\, dt\Big], \tag{6}$$

$$U_h(x, A) \triangleq \mathsf{E}_x\Big[\int_0^\infty \exp\Big\{-\int_0^t h(\Phi_s)\, ds\Big\} \mathbf{1}_A(\Phi_t)\, dt\Big]. \tag{7}$$

We have $R_h(x, f) = U_h(x, hf)$ and $R_h = R_\alpha$ when $h \equiv \alpha$. A key use of this generalized resolvent occurs when h is taken as the indicator function of a measurable set B: in this case we write $U_B \triangleq U_{\mathbf{1}_B}$, and $R_B \triangleq R_{\mathbf{1}_B}$.

The probabilistic construction of U_h and R_h enables us to write down analogues of (5), (4) for general h. On an enlarged probability space, define a randomized, possibly infinite valued, stopping time T_h by

$$\mathsf{P}_x\{T_h \in [t, t + \Delta] \mid T_h \ge t, \mathcal{F}_\infty^\Phi\} = \Delta h(\Phi_t) + o(\Delta). \tag{8}$$

In the special case where $h \equiv 1$, T_h is independent of $\mathbf{\Phi}$ and possesses a standard exponential distribution. In general, the distribution of T_h is exponential in nature, but the rate of jump at time t, instead of being constant, is modulated by the value of $h(\Phi_t)$.

With T_h so defined we have

$$R_h(x,f) = \mathsf{E}_x\Big[f(\Phi_{T_h})\mathbf{1}\{T_h < \infty\}\Big] \tag{9}$$

$$U_h(x,f) = \mathsf{E}_x\Big[\int_0^{T_h} f(\Phi_s)ds\Big] \tag{10}$$

whenever the right hand side is meaningful. This gives a recurrence condition for finiteness of T_h, shown initially in [5].

Theorem 2.1 *The following relation holds:*

$$R_h(x,\mathsf{X}) = U_h(x,h) = 1 - \mathsf{E}_x\Big[\exp\Big\{-\int_0^\infty h(\Phi_s)\,ds\Big\}\Big]. \tag{11}$$

Hence $R_h(x,\mathsf{X}) = 1$ if and only if $\mathsf{P}_x\{\int_0^\infty h(\Phi_s)\,ds = \infty\} = 1$.

Proof Note that by (9) the kernel R_h is probabilistic if and only if $T_h < \infty$ a.s. $[\mathsf{P}_*]$. We then have

$$1 = R_h(x,\mathsf{X}) = U_h(x,h) = \mathsf{E}_x\Big[\int_0^\infty \exp\Big(-\int_0^t h(\Phi_s)ds\Big)h(\Phi_t)\Big]\,dt.$$

From the change of variables $u = \int_0^t h(\Phi_s)ds$, $du = h(\Phi_t)dt$, we obtain the equality

$$\int_0^\infty \exp\Big\{-\int_0^t h(\Phi_s)\,ds\Big\}h(\Phi_t)\,dt = 1 - \exp\Big\{-\int_0^\infty h(\Phi_s)\,ds\Big\},$$

and the equality (11) follows from the definitions of R_h and U_h.

□

There is an analogue of the resolvent equation [19] for the generalized resolvent kernels U_h: for a proof see [13].

Theorem 2.2 *Let $h \ge k \ge 0$. Then U_h and U_k satisfy the generalized resolvent equation:*

$$U_k = U_h + U_h I_{(h-k)} U_k = U_h + U_k I_{(h-k)} U_h. \tag{12}$$

□

We will apply the identities in Theorem 2.3 to connect the probabilistic structure of the R-chain, the kernel U_B and the underlying Markov process.

Theorem 2.3 *For all $x \in \mathsf{X}$, $B, A \in \mathcal{B}$ we have*

(i) *$\check{G}_B(x,A) = U_B(x,A)$;*

(ii) *$\check{L}(x,B) = 1 - \mathsf{E}_x[\exp(-\eta_B)]$;*

(iii) *For all $B \in \mathcal{B}$,*

$$\lim_{t\to\infty} \check{L}(\Phi_t, B) = \lim_{t\to\infty} \mathsf{E}_{\Phi_t}[1 - \exp(-\eta_B)] = \mathbf{1}\{\eta_B = \infty\} \text{ a.s. } [\mathsf{P}_*].$$

Proof To prove (i) we apply the second form of the identity (12) with $h \equiv 1$ and $k = \mathbf{1}_B$ so that

$$U_B = U_B \mathbf{1}_{B^c} R + R.$$

If we define the n-step taboo probabilities as usual for the R-chain by ${}_B R^n \triangleq [R\mathbf{1}_{B^c}]^{n-1} R$ (see [20]) then by repeated substitution we see that, for all $n \geq 1$,

$$\begin{aligned} U_B &= U_B \mathbf{1}_{B^c} R \mathbf{1}_{B^c} R + R \mathbf{1}_{B^c} R + R \\ &= U_B \mathbf{1}_{B^c} ({}_B R)^n + \sum_{k=1}^{n} ({}_B R)^k. \end{aligned}$$

Recalling that

$$({}_B R)^n (y, f) = \mathsf{E}_y[f(\check{\Phi}_n) \mathbf{1}\{\check{\tau}_B \geq n\}],$$

where $\check{\tau}_B$ is the hitting time for the R-chain, the equality above establishes the result on letting $n \to \infty$.

Result (ii) follows immediately from (i) and Theorem 2.1.

The first equality in (iii) follows from (ii). To see the second, observe that for fixed $s \leq t$, where θ^s, $s \in \mathsf{IR}_+$, is the backward shift operator on the sample space [22], we have for any initial condition $x \in \mathsf{X}$,

$$\begin{aligned} \mathsf{E}_x[\theta^s(1 - \exp(-\eta_B)) \mid \mathcal{F}_t] &\geq \mathsf{E}_x[\theta^t(1 - \exp(-\eta_B)) \mid \mathcal{F}_t] \\ &\geq \mathsf{P}\{\eta_B = \infty \mid \mathcal{F}_t\}. \end{aligned} \tag{13}$$

Applying the martingale convergence theorem gives, as $t \to \infty$,

$$\begin{aligned} \mathsf{E}_x[\theta^s(1 - \exp(-\eta_B)) \mid \mathcal{F}_t] &\to \theta^s(1 - \exp(-\eta_B)), \\ \mathsf{P}\{\eta_B = \infty \mid \mathcal{F}_t\} &\to \mathbf{1}\{\eta_B = \infty\}. \end{aligned}$$

By the Markov property, $\mathsf{E}_x[\theta^t(1 - \exp(-\eta_B)) \mid \mathcal{F}_t] = \mathsf{E}_{\Phi_t}[(1 - \exp(-\eta_B))]$, so that letting $t \to \infty$ in (13),

$$\begin{aligned} \theta^s(1 - \exp(-\eta_B)) &\geq \limsup_{t\to\infty} \mathsf{E}_{\Phi_t}[(1 - \exp(-\eta_B))] \\ &\geq \liminf_{t\to\infty} \mathsf{E}_{\Phi_t}[(1 - \exp(-\eta_B))] \\ &\geq \mathbf{1}\{\eta_B = \infty\}. \end{aligned}$$

But as $s \to \infty$, we have

$$\theta^s(1 - \exp(-\eta_B)) \to \mathbf{1}\{\eta_B = \infty\},$$

which establishes the result.

□

Theorem 2.3 (i) is due to Neveu [19], (ii) is due to [5], and (iii) is new. As an immediate application of Theorem 2.3 (ii) and the definitions, we have that $\mathbf{\Phi}$ is φ-irreducible if and only if the R-chain is φ-irreducible. As a substantially more important use of Theorem 2.3 we now show that the definition of Harris recurrence is equivalent to a (formally) much weaker, and more useful, criterion.

Theorem 2.4 *The Markov chain $\mathbf{\Phi}$ is Harris recurrent if and only if for some σ-finite measure μ, $\{\tau_A < \infty\}$ a.s. $[\mathsf{P}_*]$ whenever $\mu\{A\} > 0$.*

Proof The necessity follows from the definition of Harris recurrence. To prove sufficiency set $\varphi = \mu R$. We will prove that if $\mathsf{P}_x\{\tau_A < \infty\} \equiv 1$ for any set A of μ-positive measure, then $\mathsf{P}_x\{\eta_B = \infty\} \equiv 1$ for any set B of φ-positive measure, and hence that $\mathbf{\Phi}$ is Harris recurrent.

Observe that for any measurable set B we have, by the definitions,

$$\varphi\{B\} > 0 \Longleftrightarrow \int \mu(dx)\mathsf{E}_x[1 - \exp(\eta_B)] > 0. \tag{14}$$

From the assumption of the theorem

$$\int f\, d\mu > 0 \Longrightarrow \limsup_{t\to\infty} f(\Phi_t) > 0, \qquad \text{a.s. } [\mathsf{P}_*] \tag{15}$$

for any bounded measurable function $f\colon \mathsf{X} \to \mathsf{IR}$. Suppose that $\varphi\{B\} > 0$ so that by (14) and (15),

$$\limsup_{t\to\infty} \mathsf{E}_{\Phi_t}[1 - \exp(\eta_B)] > 0 \qquad \text{a.s. } [\mathsf{P}_*]$$

Then by Theorem 2.3 (iii), $\mathsf{P}_x\{\eta_B = \infty\} = 1$ for all x, which is the desired conclusion. □

The conditions of Theorem 2.4 are much easier to verify than those of the original definition of Harris recurrence. A simple counter-example shows that the measures μ and φ do not coincide in general. Consider the deterministic uniform motion on the unit circle S^1 in the complex plane described by the equation

$$\Phi_t = e^{2\pi i t}\Phi_0 \qquad t \geq 0. \tag{16}$$

For this chain, we can take $\mu = \delta_x$ for any x, since the process reaches every point, and in unit time; but since μ is singular with respect to Lebesgue measure, it will not allocate positive mass to sets visited for an infinite amount of time by the chain.

A version of Theorem 2.4 is also given in [9] with a substantially more complicated proof.

3 Sampled Chains, Petite Sets & Harris Recurrence

The second generalization of resovents involves moving from the exponential to a more general, but independent, sampling distribution.

Suppose that a is a general probability on IR_+, and define the Markov transition function K_a as

$$K_a \triangleq \int P^t\, a(dt). \tag{17}$$

If a is the increment distribution of the undelayed renewal process $\{T(k)\}$, then K_a is the transition function for the Markov chain $\{\Phi_{T(k)}\}$. In the special case where a is an exponential distribution with unit mean, the transition function K_a is the resolvent R of the process.

We now demonstrate the connections between the recurrence of $\mathbf{\Phi}$ and the embedded chain $\{\Phi_{T(k)}\}$.

Theorem 3.1 *Suppose that a is a general probability on IR_+. If the K_a-chain is Harris recurrent, then so is the process $\mathbf{\Phi}$; and then the K_a-chain is positive Harris recurrent if and only if the process $\mathbf{\Phi}$ is positive Harris recurrent.*

Proof The first result is a trivial consequence of Theorem 2.4, since the hitting time on a set A by the process occurs no later than the first hitting time in the process sampled with distribution a.

If the process is then positive Harris, it is simple to observe that an invariant measure satisfying $\pi = \pi P^s$ for all s must satisfy $\pi = \pi K_a$ also. To see the converse, we observe first that by the Chapman Kolmogorov equations we have, for any sampling distribution a and any s, the commutative identity

$$\begin{aligned} &\int \Big[\int P^t(x, dy) a(dt)\Big] P^s(y, A) \\ &= \int P^{t+s}(x, A) a(dt) \\ &= \int P^s(x, dy) \Big[\int P^t(y, A) a(dt)\Big]. \end{aligned}$$

Let π be the unique invariant probability measure for K_a. Using this identity

$$\begin{aligned} &\int \pi(dy) P^s(y, A) \\ &= \int \Big[\int \pi(dx) K_a(x, dy)\Big] P^s(y, A) \\ &= \int \Big[\int \pi(dx) P^s(x, dy)\Big] K_a(y, A). \end{aligned}$$

This tells us that πP^s is an invariant probability measure for K_a and since π is unique, we have $\pi = \pi P^s$; thus by definition $\mathbf{\Phi}$ is positive Harris.

□

The first of these results extends Theorem 2.2 of [23], which required additional conditions on the process or on a due to the use of the initial form of Harris recurrence in terms of η_A. In the special case of resolvents the second result is proved in [2], but with a rather more difficult proof.

Harris recurrence of the process can under certain conditions also imply that the sampled chain is recurrent: see [23] for details. However, the clock process (16) shows that this is not always true: we merely need a concentrated on $\mathbb{Z}_+$ for K_a to have an uncountable collection of absorbing sets. For more detailed analysis of this situation see [17].

Sampled chains for which a possesses a bounded density will prove to be more tractable than general sampled chains. The following connects such chains to the resolvent.

Proposition 3.2 *Suppose that the distribution a on $(0, \infty)$ possesses a bounded density with respect to Lebesgue measure. Then there exists a constant $0 < M < \infty$ such that*

$$K_a(x, B) \le M \check{L}(x, B) \qquad \forall x \in \mathsf{X},\ B \in \mathcal{B}.$$

Proof As in the proof of Theorem 3.1 of [23], we may show that if a possesses a bounded density f on $(0, \infty)$ then there exists a constant $M < \infty$ such that

$$\int_0^\infty \mathbf{1}\{\Phi_t \in B\} f(t)\, dt \le M(1 - \exp(-\eta_B)), \qquad B \in \mathcal{B}$$

Taking expectations and applying Theorem 2.3 (ii) we see that

$$K_a(x,B) \le M\mathsf{E}_x[1-\exp(-\eta_B)] = M\check{L}(x,B) \qquad \forall x \in \mathsf{X}$$

which proves the result.

□

We now introduce the class of *petite sets*. These will be seen to play the same role as the *small sets* of [20]. In particular, we will show that they are test sets or "status sets" [24] for Harris recurrence, as they are in discrete time (see [16, 15]).

A non-empty set $A \in \mathcal{B}$ is called φ_a*-petite* if φ_a is a non-trivial measure on $\mathcal{B}$ and a is a probability distribution on $(0,\infty)$ satisfying,

$$K_a(x,\cdot) \ge \varphi_a(\,\cdot\,)$$

for all $x \in A$. The distribution a is called the *sampling distribution* for the petite set A.

Harris recurrence may be characterized by the finiteness of the hitting time to a single petite set.

Theorem 3.3 *If C is petite, and $\mathsf{P}_x\{\tau_C < \infty\} = 1$ for all $x \in \mathsf{X}$, then $\mathbf{\Phi}$ is Harris recurrent.*

Proof We have that C is φ_a-petite. Let b denote a distribution on $\mathbb{R}_+$ with a bounded, continuous density, and define $\varphi = \varphi_a K_b$, so that

$$K_{a*b}\,(x,B) = K_a K_b\,(x,B) \ge \varphi_a K_b\,\{B\} = \varphi\{B\}, \qquad x \in C.$$

To prove the theorem we demonstrate that

$$\varphi\{B\} > 0 \Longrightarrow \eta_B = \infty \qquad \text{a.s. } [\mathsf{P}_*] \tag{18}$$

Since $a*b$ has a bounded continuous density, by Proposition 3.2, there exists $M < \infty$ for which

$$K_{a*b}\,(x,B) \le M\mathsf{E}_x[1-\exp(\eta_B)] \qquad \forall x \in \mathsf{X},\ B \in \mathcal{B}.$$

Hence if $\varphi\{B\} > 0$ then,

$$\mathsf{E}_x[1-\exp(\eta_B)] \ge \frac{1}{M}\varphi\{B\} > 0 \qquad x \in C. \tag{19}$$

The assumption of the theorem is

$$\limsup_{t\to\infty} \mathbf{1}(\Phi_t \in C) = 1 \qquad a.s.\ [\mathsf{P}_*],$$

and hence from (19) and Theorem 2.3 (iii),

$$\mathbf{1}\{\eta_B = \infty\} = \limsup_{t\to\infty} \mathsf{E}_{\Phi_t}[1-\exp(\eta_B)] \ge \frac{\varphi\{B\}}{M} \qquad a.s.\ [\mathsf{P}_*]$$

which shows that (18) holds.

□

This theorem is not vacuous: a φ-irreducible process admits a large number of petite sets.

Proposition 3.4 *If $\mathbf{\Phi}$ is φ-irreducible then*

(i) *the state space may be expressed as the union of a countable collection of petite sets, and hence in particular a closed petite set in $\mathcal{B}^+$ always exists.*

(ii) *If C is ν_a-petite then for any $\alpha > 0$, there exists an integer $m \geq 1$ and a maximal irreducibility measure ψ_m such that for any $B \in \mathcal{B}$ and any $x \in C$,*

$$R_\alpha^m(x, B) \geq \psi_m(B).$$

Proof (i) We know that the R-chain and hence every R_α-chain is φ-irreducible for some φ. Applying Theorem 2.1 of [20], it follows that the transition function K_a used in the defining relation for petite sets may be taken to be R_α^n for some $n \in \mathbb{Z}_+$; and there is a countable cover by petite sets for the R_α-chain by Proposition 5.13 of [20].

Hence there is a petite set in $\mathcal{B}^+$, and by the regularity of the measure ψ and the fact that subsets of petite sets are petite, a closed petite set in $\mathcal{B}^+$.

(ii) Supposing that C is ν_a-petite, by the preceding discussion there exists a set C_n which is ν_n-small for the R_α-chain with $\nu_n\{C_n\} > 0$: that is, in the nomenclature of [20], $R_\alpha^n(x, \cdot\,) \geq \nu_n(\,\cdot\,)$ for $x \in C_n$. The state space for the R_α-chain can be written as the union of small sets, and hence we may assume that $C_n \in \mathcal{B}^+$, from which it follows that ν_n is an irreducibility measure. We now modify ν_n to obtain a maximal irreducibility measure. Choose t_0 so large that

$$\int_0^{t_0} P^t(x, C_n)\, a(dt) = \delta > 0, \qquad x \in C,$$

and hence

$$\int_0^{t_0} P^t R_\alpha^n(x, \cdot\,)\, a(dt) \geq \delta \nu_n(\,\cdot\,), \qquad x \in C \tag{20}$$

We now use the simple estimate $P^t R_\alpha \leq e^{\alpha t} R_\alpha$ to obtain by (20),

$$R_\alpha^n(x, \cdot\,) \geq e^{-\alpha t_0} \delta \nu_n(\,\cdot\,), \qquad x \in C.$$

Applying R_α to both sides of this inequality gives for any $x \in C$

$$R_\alpha^{n+1}(x, \cdot\,) \geq e^{-\alpha t_0} \delta \nu_n R_\alpha(\,\cdot\,)$$

It is easy to see that $\nu_n R_\alpha$ is a maximal irreducibility measure, and hence this completes the proof with $m = n + 1$ and $\psi_m = e^{-\alpha t_0} \delta \nu_n R_\alpha$.

□

Even without irreducibility, a countable covering of the state space by petite sets may often be constructed. For example, a diffusion whose generator is hypoelliptic has the property that its resolvent has the strong Feller property. From this it follows that the state space admits a covering by open petite sets [15].

4 Finiteness of $\pi(f)$

We have shown in the previous section that petite sets, like single points for discrete state space processes, may be used to classify a Markov process as Harris recurrent. They have a similar role in the classification of positive Harris chains, and the goal of this section is to prove Theorem 1.2 where we defined that relationship.

We will fix throughout this section a measurable function $f \geq 1$, and assume that $\mathbf{\Phi}$ is Harris recurrent with invariant measure π. The assumption of Theorem 1.2 (b)(i), which we wish to show is equivalent to $\pi(f) < \infty$, is

There exists a closed petite set C such that for some $\delta > 0$

($\mathcal{R}(\delta)$)
$$\sup_{x \in C} G_C(x, f; \delta) < \infty$$

In order to connect this condition with the finiteness of $\pi(f)$ we use the fact that π is also the invariant measure for $\breve{\mathbf{\Phi}}$, and hence from Proposition 5.9 of [20] and Theorem 2.3 has the representation

$$\pi(f) = \int_C \pi(dy) \breve{G}_C(x, f) = \int_C \pi(dy) U_C(x, f). \tag{21}$$

Proving Theorem 1.2 is thus a matter of obtaining appropriate bounds between the kernels $G_C(x, f; \delta)$ and $U_C(x, f)$ for sets with $\pi(C) < \infty$, such as ([16], Theorem 5.2) petite sets: elucidating these and related results occupies most of this section.

For any x and $f \geq 1$ it is obvious that $G_C(x, f; r)$ is an increasing function of r. The following result gives conditions under which the rate of growth may be bounded.

Lemma 4.1 *Suppose that $G_C(x, f; r)$ is bounded on the closed set C for some $r > 0$. Then there exists $k < \infty$ such that for any $t \geq 0$,*

$$G_C(x, f; t) \leq G_C(x, f; r) + kt, \qquad x \in \mathsf{X}.$$

Hence in particular ($\mathcal{R}(\delta)$) implies ($\mathcal{R}(t)$) for every $t > 0$.

Proof Let $\tau_C^k(r)$ denote the kth iterate of $\tau_c(r)$ defined inductively by

$$\tau_C^0(r) = 0 \text{ and } \tau_C^{n+1}(r) = \tau_C^n(r) + \theta^{\tau_C^n(r)} \tau_C(r). \tag{22}$$

We have the simple bound, valid for any positive integer n,

$$G_C(x, f; nr) \leq \mathsf{E}_x \Big[\sum_{i=0}^{n-1} \theta^{\tau_C^i(r)} \int_0^{\tau_C(r)} f(\Phi_s)\, ds \Big]$$

and hence by the strong Markov property and the assumption that the set C is closed,

$$G_C(x, f; nr) \leq G_C(x, f; r) + (n-1) \sup_{y \in C} G_C(y, f; r)$$

This bound together with the fact that $G_C(x, f; t)$ is an increasing function of t completes the proof.

□

Following discrete time usage [20, 15] a non-empty set $C \in \mathcal{B}$ is called *f-regular* if

$$G_B(x, f; \delta) = \mathsf{E}_x\Big[\int_0^{\tau_B(\delta)} f(\Phi_t)\, dt\Big]$$

is bounded on C for any $\delta > 0$ and any $B \in \mathcal{B}^+$.

The next key result shows that the "self regularity" property $\mathcal{R}(\delta)$ for C closed and petite actually implies f-regularity.

Proposition 4.2 *Consider a Harris recurrent Markov process $\mathbf{\Phi}$ such that $\mathcal{R}(\delta)$ holds. Then C is f-regular.*

Proof Let $B \in \mathcal{B}^+$, and choose $t > 0$ so large that for some $\varepsilon > 0$, $\mathsf{P}_x\{\tau_B \le t\} \ge \varepsilon$ for $x \in C$. This is possible since C is petite. From Lemma 4.1, we have $G_C(x, f; t)$ bounded on C no matter how large we choose t.

Let $\sigma(k) \triangleq \tau_C^k(t)$, the kth iterate of $\tau_C(t)$ defined in (22). Conditioning on $\mathcal{F}_{\sigma(k-1)}$ and using induction gives

$$\mathsf{P}_x\{\tau_B > \sigma(k)\} \le (1-\varepsilon)^k, \qquad k \in \mathbb{Z}_+,\ x \in C. \tag{23}$$

We will consider the bound

$$\mathsf{E}_x\Big[\int_0^{\tau_B} f(\Phi_t)\, dt\Big] \le \lim_{n\to\infty} S_n(x), \tag{24}$$

where

$$S_n(x) \triangleq \sum_{k=0}^{n} \mathsf{E}_x\Big[\int_0^{\sigma(k+1)} f(\Phi_t)\, dt \mathbf{1}\{\sigma(k) < \tau_B \le \sigma(k+1)\}\Big]$$

We will exhibit a contractive property for the sequence $\{S_n(x)\}$ which implies that the sequence is uniformly bounded, and hence that $\mathsf{E}_x[\int_0^{\tau_B} f(\Phi_t)\, dt]$ is bounded on C. This together with the estimate

$$\begin{aligned} &\int_0^{\tau_B(t)} f(\Phi_t)\, dt \\ &\quad \le \int_0^{\tau_C(t)} f(\Phi_t)\, dt + \theta^{\tau_C(t)} \int_0^{\tau_B} f(\Phi_t)\, dt \end{aligned}$$

and the strong Markov property will complete the proof of the proposition.

We have for any $n \ge 1$,

$$\begin{aligned} S_n(x) &= \mathsf{E}_x\Big[\int_0^{\sigma(1)} f(\Phi_t)\, dt \mathbf{1}\{0 < \tau_B \le \sigma(1)\}\Big] \\ &\quad + \sum_{k=1}^{n} \mathsf{E}_x\Big[\int_0^{\sigma(1)} f(\Phi_t)\, dt \mathbf{1}\{\sigma(1) < \tau_B\}\theta^{\sigma(1)}\mathbf{1}_{\mathcal{B}(k)}\Big] \\ &\quad + \sum_{k=1}^{n} \mathsf{E}_x\Big[\int_{\sigma(1)}^{\sigma(1)+\theta^{\sigma(1)}\sigma(k)} f(\Phi_t)\, dt \mathbf{1}\{\sigma(1) < \tau_B\}\theta^{\sigma(1)}\mathbf{1}_{\mathcal{B}(k)}\Big] \end{aligned} \tag{25}$$

where $\mathcal{B}(k) \triangleq \{\sigma(k-1) < \tau_B \le \sigma(k)\}$.

By conditioning at time $\sigma(1)$ the first two lines on the RHS of this inequality may be bounded by

$$\mathsf{E}_x\Big[\int_0^{\sigma(1)} f(\Phi_t)\,dt\Big]\Big(1+\sum_{k=1}^{\infty}\sup_{x\in C}\mathsf{P}_x\{\tau_B > \sigma(k-1)\}\Big) \le (1+\varepsilon^{-1})\mathsf{E}_x\Big[\int_0^{\sigma(1)} f(\Phi_t)\,dt\Big] \qquad (26)$$

where ε is defined above equation (23).

The last line on the RHS of (25) is bounded in a similar way: by conditioning at time $\sigma(1)$ we may bound this term by

$$\mathsf{P}_x\{\sigma(1) < \tau_B\}\sup_{y\in C}\sum_{k=1}^{n}\mathsf{E}_y\Big[\int_0^{\sigma(k)} f(\Phi_t)\,dt\mathbf{1}\{\sigma(k-1) < \tau_B \le \sigma(k)\}\Big],$$

which is identical to $\mathsf{P}_x\{\sigma(1) < \tau_B\}\sup_{y\in C} S_{n-1}(y)$. This together with (23), (25), and (26) implies that

$$\sup_{x\in C} S_n(x) \le (1-\varepsilon)\sup_{x\in C} S_{n-1}(x) + \sup_{x\in C}(1+\varepsilon^{-1})\mathsf{E}_x\Big[\int_0^{\sigma(1)} f(\Phi_t)\,dt\Big].$$

Since by definition $\mathsf{E}_x\Big[\int_0^{\sigma(1)} f(\Phi_t)\,dt\Big] = G_C(x,f;t)$ this shows that $\sup\{S_n(x) : n\in\mathbb{Z}_+,\ x\in C\} < \infty$, and hence $\mathsf{E}_x[\int_0^{\tau_B} f(\Phi_s)\,ds]$ is bounded on C, which completes the proof. □

Sets which are f-regular are not hard to locate. We have

Proposition 4.3 **(i)** *If C is f-regular then C is petite.*

(ii) *Suppose that C is closed and f-regular. Then for each n the set*

$$C_n = \{x : G_C(x,f) \le n\}$$

is f-regular whenever it is non-empty, and the union of the $\{C_n\}$ is full. Hence, in particular, there exists a closed petite f-regular set.

Proof **(i)** Let C be f-regular, and let $A \in \mathcal{B}^+$ be any other closed petite set. For any positive measurable function g we have the bound, valid for any $x \in \mathsf{X}$,

$$\begin{aligned} R(x,g) &\ge \mathsf{E}_x\Big[\int_{\tau_A}^{\infty} g(\Phi_s)e^{-s}\,ds\Big] \\ &= \mathsf{E}_x[e^{-\tau_A}R(\Phi_{\tau_A},g)] \end{aligned}$$

Taking $g = \sum 2^{-n}R^{n-1}(\,\cdot\,,A)$ any applying Jensen's inequality we see that for any x

$$\sum_{n=1}^{\infty} 2^{-n}R^n(x,A) \ge \exp(\mathsf{E}_x[-\tau_A])\inf_{x\in A}\sum_{n=1}^{\infty} 2^{-n}R^n(x,A)$$

Since $A \in \mathcal{B}^+$ and A is petite, it follows that the infimum is strictly positive. By regularity the exponential is bounded from below on C, and hence

$$\inf_{x \in C} \sum_{n=1}^{\infty} 2^{-n} R^n(x, A) > 0$$

which shows that C is petite from Lemma 3.1 of [16].

(ii) Since C is f-regular it is ψ_a-petite for a distribution a with finite mean $m(a)$ by Proposition 3.4(ii). By Lemma 4.1 we have for some $k < \infty$, and any $t \geq 0$

$$P^t G_C\,(x, f; \delta) \leq G_C(x, f; \delta + t) \leq G_C(x, f; \delta) + (\delta + t)k$$

and hence for any $x \in C$,

$$\begin{aligned} \int \psi_a(dy) G_C(y, f; \delta) &\leq K_a G_C\,(x, f; \delta) \\ &= \int_0^{\infty} P^t G_C\,(x, f; \delta)\, a(dt) \\ &\leq G_C\,(x, f; \delta) + k[m(a) + \delta] < \infty \end{aligned}$$

Hence also $\int \psi_a(dy) G_C(y, f; 0) < \infty$, and since ψ_a is maximal this shows that the set of all $x \in \mathsf{X}$ for which $G_C(x, f; 0)$ is finite is full.

That C_n is f-regular follows immediately from the inequality

$$\int_0^{\tau_B(\delta)} f(\Phi_s)\, ds \leq \int_0^{\tau_C} f(\Phi_s)\, ds + \theta^{\tau_C} \int_0^{\tau_B(\delta)} f(\Phi_s)\, ds$$

and the strong Markov property. ◻

We can now connect f-regularity of a set C with the condition that $\sup_{x \in C} U_C(x, f) < \infty$, which is what we need in order to use (21).

Proposition 4.4 *For a Harris recurrent Markov process* $\mathbf{\Phi}$ *the following implications hold for a set* $C \in \mathcal{B}$*:*

(i) *If C is petite and* $\sup_{x \in C} U_C(x, f) < \infty$ *then C is f-regular and $C \in \mathcal{B}^+$.*

(ii) *If C is closed and f-regular, and if $C \in \mathcal{B}^+$, then* $\sup_{x \in C} U_C(x, f) < \infty$ *and C is petite.*

Proof **(i)** It follows from Proposition 3.4 and Theorem 2.3 (i), together with Theorem 11.3.14 of [15] that C is f-regular for the R-chain. That is, $U_B(x, f)$ is bounded on C for any $B \in \mathcal{B}^+$.

Hence for some constant M_B and all $x \in C$,

$$M_B \;\geq\; \mathsf{E}_x\Big[\int_0^{\infty} \exp\Big(-\int_0^t \mathbf{1}\{\Phi_s \in B\}\, ds\Big) f(\Phi_t)\, dt\Big]$$

$$
\begin{aligned}
&\geq \mathsf{E}_x\Big[\int_0^{\tau_B(\delta)} \exp\Big(-\int_0^t \mathbf{1}\{\Phi_s\in B\}\,ds\Big) f(\Phi_t)\,dt\Big] \\
&\geq \mathsf{E}_x\Big[\int_0^{\delta} e^{-\delta} f(\Phi_t)\,dt\Big] + \mathsf{E}_x\Big[\int_\delta^{\tau_B(\delta)} e^{-\delta}\exp\Big(-\int_\delta^t \mathbf{1}\{\Phi_s\in B\}\,ds\Big) f(\Phi_t)\,dt\Big] \\
&= \mathsf{E}_x\Big[\int_0^{\delta} e^{-\delta} f(\Phi_t)\,dt\Big] + \mathsf{E}_x\Big[\int_\delta^{\tau_B(\delta)} e^{-\delta} f(\Phi_t)\,dt\Big] \\
&= e^{-\delta} G_B^{\delta}(x,f)
\end{aligned}
$$

which shows that (i) holds.

(ii) By Proposition 4.3 the set C is ψ_a-petite with $\psi_a(C) > 0$ so that $\mathsf{P}_x\{\eta_C = \infty\} = \delta$ is bounded from below for $x \in C$. Hence, again using the fact that C is petite, there exists $T > 0$ and $\beta < 1$ such that

$$
\begin{aligned}
\sup_{x\in C} \mathsf{E}_x\Big[\exp\Big(-\int_0^{\tau_C(T)} \mathbf{1}\{\Phi_s\in C\}\,ds\Big)\Big] &\leq \sup_{x\in C} \mathsf{E}_x\Big[\exp\Big(-\int_0^{T} \mathbf{1}\{\Phi_s\in C\}\,ds\Big)\Big] \\
&= \beta,
\end{aligned} \tag{27}
$$

and by f-regularity and ψ-positivity of the set C we have for any T,

$$
\sup_{x\in C} \mathsf{E}_x\Big[\int_0^{\tau_C(T)} f(\Phi_t)\,dt\Big] < \infty \tag{28}
$$

To avoid dealing with possibly infinite terms, let $h\colon \mathsf{X} \to \mathbb{R}_+$ be a measurable function for which

$$
h(x) \geq \mathbf{1}_C(x) \qquad \text{and} \qquad \inf_{x\in \mathsf{X}} h(x) > 0,
$$

and fix $N \geq 1$. Then we may approximate as follows:

$$
\begin{aligned}
&\mathsf{E}_x\Big[\int_0^{\infty} \exp\Big(-\int_0^t h(\Phi_s)\,ds\Big) N \wedge f(\Phi_t)\,dt\Big] \\
&\quad\leq \mathsf{E}_x\Big[\int_0^{\tau_C(T)} f(\Phi_t)\,dt\Big] \\
&\qquad + \mathsf{E}_x\Big[\int_{\tau_C(T)}^{\infty} \exp\Big(-\int_0^t h(\Phi_s)\,ds\Big) N \wedge f(\Phi_t)\,dt\Big]
\end{aligned} \tag{29}
$$

and each of these terms is finite since h is strictly positive. Once we obtain the desired bounds we will let $h \downarrow \mathbf{1}_C$ and $N \uparrow \infty$.

The second term on the right hand side of (29) can be bounded as follows, using the strong Markov property:

$$
\begin{aligned}
&\mathsf{E}_x\Big[\int_{\tau_C(T)}^{\infty} \exp\Big(-\int_0^t h(\Phi_s)\,ds\Big) N \wedge f(\Phi_t)\,dt\Big] \\
&\quad= \mathsf{E}_x\Big[\int_{\tau_C(T)}^{\infty} \exp\Big(-\int_0^{\tau_C(T)} h(\Phi_s)\,ds\Big) \exp\Big(-\int_{\tau_C(T)}^t h(\Phi_s)\,ds\Big) N \wedge f(\Phi_t)\,dt\Big]
\end{aligned}
$$

$$= \mathsf{E}_x\Big[\exp\Big(-\int_0^{\tau_C(T)} h(\Phi_s)\,ds\Big)\mathsf{E}_{\Phi_{\tau_C(T)}}\Big[\int_0^\infty \exp\Big(-\int_0^r h(\Phi_s)\,ds\Big)N\wedge f(\Phi_r)\,dr\Big]\Big]$$

$$\leq \mathsf{E}_x\Big[\exp\Big(-\int_0^{\tau_C(T)} h(\Phi_s)\,ds\Big)\Big]\sup_{y\in C}\mathsf{E}_y\Big[\int_0^\infty \exp\Big(-\int_0^r h(\Phi_s)\,ds\Big)N\wedge f(\Phi_r)\,dr\Big]$$

where in the last inequality we have used the assumption that C is closed, so that $\Phi_{\tau_C(T)} \in C$. Hence, using (27), we have from this and (29),

$$\sup_{x\in C}\mathsf{E}_x\Big[\int_0^\infty \exp\Big(-\int_0^t h(\Phi_s)\,ds\Big)N\wedge f(\Phi_t)\,dt\Big] \leq \sup_{x\in C}\frac{\mathsf{E}_x\Big[\int_0^{\tau_C(T)} f(\Phi_t)\,dt\Big]}{1-\beta} < \infty \tag{30}$$

Letting $h \downarrow \mathbf{1}_C$ and $N \uparrow \infty$ we see that $\sup_{x\in C} U_C(x,f) < \infty$ by monotone convergence, which proves the result.

□

Proof of Theorem 1.2

To see that (i) $\Longrightarrow$ (ii), observe that if $\mathcal{R}(\delta)$ holds then we have from Proposition 4.3 that there exists an f-regular closed set $C \in \mathcal{B}^+$. Then $U_C(x,f)$ is bounded and C is petite by Proposition 4.4 (ii). Hence $\pi(f)$ is finite from (21).

Conversely, if $\pi(f) < \infty$ we have from (21), Theorem 10.4.6 of [15] and Theorem 2.3 (i) that $U_C(x,f)$ is bounded on some petite set C, and that this implies (i) follows from Proposition 4.4 (i).

□

5 Comments and Extensions

The emphasis of this paper has been on the characterization of Harris recurrence and its refinements. From an applications point of view, it is important to obtain conditions under which bounds such as ($\mathcal{R}(\delta)$) hold.

In discrete time such bounds follow from generalized forms of Foster-Lyapunov criteria [15] applied to the one-step transition laws P. For processes in continuous time, it is more natural to construct bounds by considering the infinitesimal generator for the process. If a drift towards the "center" of the state space can be established using the generator, then an application of Dynkin's formula immediately gives bounds such as ($\mathcal{R}(\delta)$) (see [18]). Even if the generator cannot be easily analyzed, a stability proof often yields as a by-product bounds such as ($\mathcal{R}(\delta)$). One such instance is the stability analysis of generalized Jackson networks given in [14, 6]. The bound ($\mathcal{R}(\delta)$) is verified in these papers using a "long time window" stability proof, since it is not possible to establish an infinitesimal drift for the Markov process under consideration in these applications.

We have not discussed here the rich ergodic theory which is a consequence of these results, and which is often the reason for interest in Harris recurrence. Several ergodic theorems which are based upon the results presented here may be found in [7, 17, 18], and we conclude with two significant applications of the results proved above.

A Markov process $\mathbf{\Phi}$ is called *ergodic* if an invariant probability π exists and

$$\lim_{t\to\infty} \|P^t(x,\,\cdot\,) - \pi\| = 0, \qquad x \in \mathsf{X},$$

where $\|\cdot\|$ denotes the total variation norm. Unlike their discrete time counterparts, Harris recurrent processes do not always possess the ergodic property even when π is finite, and additional conditions on the process are required to obtain ergodicity.

If $\mathbf{\Phi}$ is ergodic then it follows that the skeleton chain $\{\Phi_{\Delta n} : n \in \mathbb{Z}_+\}$ is also ergodic for each $\Delta > 0$. Results linking ergodic results of the process and its skeletons are well known for countable space processes. In a general setting, in [23] it is shown that under some continuity conditions (in t) on the semigroup P^t, ergodicity of the process $\mathbf{\Phi}$ follows from the ergodicity of the embedded skeletons or of the resolvent chains. The following extension of those results is taken from [17]: it indicates the value of criteria for positive Harris chains such as those we have presented above.

Theorem 5.1 *If $\mathbf{\Phi}$ is positive Harris recurrent and if some skeleton chain is irreducible, then $\mathbf{\Phi}$ is ergodic.*

By generalizing the kernel U_h further we can also obtain conditions which ensure that $P^t(x, f)$ converges to $\pi(f)$ at an exponential rate, even when the function f is unbounded. Although the kernels U_h and R_h of Section 2 have only been defined with $h \geq 0$, there is no reason why the functions h and k cannot take on negative values in the definitions of U_h and R_h, and in Theorem 2.2. This observation is crucial in the application to analysis of exponentially ergodic chains in [7], from which we take

Theorem 5.2 *For a function $V \geq 1$, suppose that there exists a closed petite set C and $\delta, \varepsilon > 0$ such that*

$$\mathsf{E}_x\Big[\int_0^{\tau_C(\delta)} e^{\varepsilon t} V(\Phi_s)\, ds\Big]$$

is everywhere finite and bounded on C. Then $\mathbf{\Phi}$ is positive Harris recurrent with invariant probability π. If a skeleton chain is irreducible, then there exists $M < \infty$, $\rho < 1$ such that for any measurable function $|f| \leq V$ and any $x \in \mathsf{X}$,

$$|P^t(x, f) - \pi(f)| \leq M \mathsf{E}_x\Big[\int_0^{\tau_C(\delta)} e^{\varepsilon t} V(\Phi_s)\, ds\Big]\rho^t, \qquad t \geq 0$$

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